Seminar on Probability and Computer Science

An Introduction to Martingales and Their Application

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Conditional Expectations and Martingales

Conditional Expectations

Definition (Conditional Expectation)
Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F} \subseteq \mathcal{A}$ be a σ-algebra over Ω. The conditional expectation of X with respect to $\mathcal{F}$ is a $\mathcal{F}$-$\mathcal{B}(\mathbb{R})$-measurable random variable $X^{\mathcal{F}} : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for which we have

$$\int_F X \, \mathrm{d}\mathbb{P} = \int_F X^{\mathcal{F}} \, \mathrm{d}\mathbb{P}_{|\mathcal{F}} \text{ for all } F \in \mathcal{F}. \tag{CE}$$

Remark
Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $Y : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ measurable. We will use the abbreviations $\mathbb{E}(X \mid Y) := \mathbb{E}(X \mid Y^{-1}(\mathcal{E}))) := X^{Y^{-1}(\mathcal{E})}$.

Existence & Uniqueness of Conditional Expectations

Theorem (Existence & Uniqueness of Conditional Expectations)
Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F} \subseteq \mathcal{A}$ be a σ-algebra over Ω.

(a) If $X^{\mathcal{F}}, Y^{\mathcal{F}} \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ satisfy

$$\mathbb{E}(X \mathbb{1}_F) = \mathbb{E}(X^{\mathcal{F}} \mathbb{1}_F) = \mathbb{E}(Y^{\mathcal{F}} \mathbb{1}_F) \text{ for all } F \in \mathcal{F},$$

then $X^{\mathcal{F}} = Y^{\mathcal{F}}$ $\mathbb{P}$-almost surely.

(b) A random variable $X^{\mathcal{F}} : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ fulfilling (CE) exists.

Proof Sketch.

- The positive and negative part of X are densities with respect to $\mathbb{P}$ and thus each induce a measure on $(\Omega, \mathcal{F})$.
- As both of these measures are absolutely continuous with respect to $\mathbb{P}_{|\mathcal{F}}$, by the Radon-Nikodym-Theorem there exist densities with respect to $\mathbb{P}_{|\mathcal{F}}$.
- Subtracting these densities, one obtains $X^{\mathcal{F}}$.
- $X^{\mathcal{F}}$ is unique as any absolute difference of two candidates is integrable and must almost surely be zero due to (CE).

□

Definition (Discrete Filtration)
Let $(\Omega, \mathcal{A})$ be a measurable space. A sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ of sub σ-fields of $\mathcal{A}$ over Ω is called a filtration over $(\Omega, \mathcal{A})$.

Definition (Discrete Martingale)
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration over $(\Omega, \mathcal{A})$. The random variables $(X_n)_{n\in\mathbb{N}}$ are called a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ $:\Longleftrightarrow$

(a) $X_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ and

(b) $\mathbb{E}\left(X_{n+1} \mid \mathcal{F}_n\right) = X_n$ $\mathbb{P}$-almost surely for all $n \in \mathbb{N}$.

A Simple Martingale

Example (Gambler's Ruin 1)

Let $(Z_n)_{n\in\mathbb{N}}$ be i.i.d. random variables such that the associated probability measure is

$$\mathbb{P}_{Z_1} = \frac{\delta_1 + \delta_{-1}}{2}\,.$$

We define

$$X_n := \sum_{j=1}^{n} Z_j \text{ for all } n \in \mathbb{N}\,.$$

$(X_n)_{n\in\mathbb{N}}$ is a martingale with respect to σ-algebra $\mathcal{F}_n := \mathcal{I}(Z_1, \ldots, Z_n)$, as

(a) $\mathbb{E}|X_n| = n\mathbb{E}|Z_1| = n$ and X_n is $\mathcal{F}_n$-$\mathcal{B}(\mathbb{R})$-measurable and

(b) $\mathbb{E}(X_{n+1} \mid \mathcal{F}_n) = \mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) + X_n = \mathbb{E}Z_{n+1} + X_n = X_n\,.$

A Simple Martingale

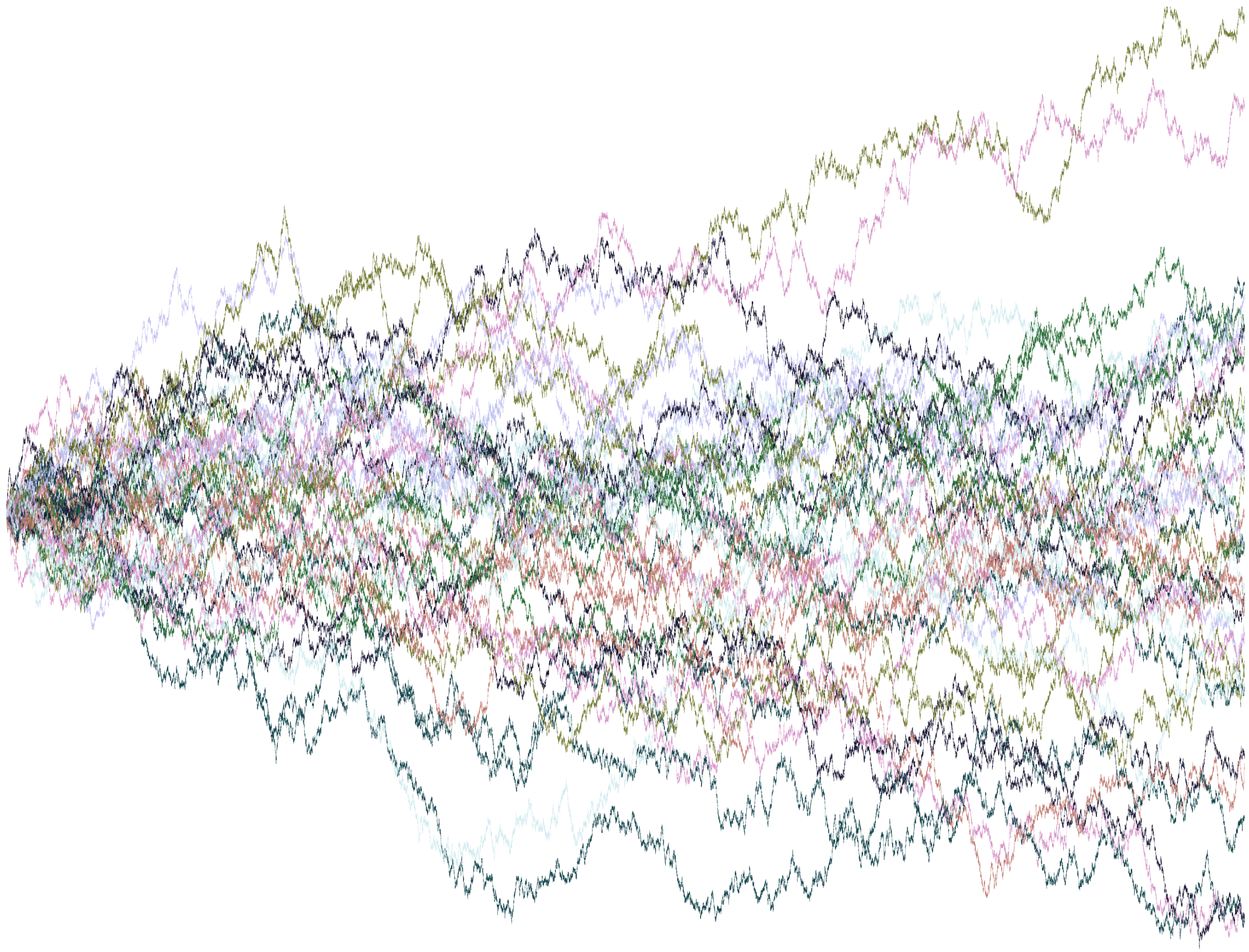

Figure 1: A realization of 50 independent copies of $(X_k)_{k \leq n}$, where $n = 7 \cdot 10^4$.

Stopping Times

The Stopping Lemma

Lemma (Stopping Lemma)
For a martingale $(X_n)_{n\in\mathbb{N}}$ *with respect to filtration* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *we have*

$$\mathbb{E}X_n = \mathbb{E}X_1 \textit{ for all } n \in \mathbb{N}\,.$$

Proof.
Due to property (b) of martingales we have

$$\begin{aligned} & \mathbb{E}\left(X_{n+1} \mid \mathcal{F}_n\right) = X_n && \mathbb{P}\text{-almost surely for all } n \in \mathbb{N} \\ \Longrightarrow\; & \mathbb{E}\left(\mathbb{E}\left(X_{n+1} \mid \mathcal{F}_n\right)\right) = \mathbb{E}X_n && \text{for all } n \in \mathbb{N} \\ \Longleftrightarrow\; & \mathbb{E}X_{n+1} = \mathbb{E}X_n && \text{for all } n \in \mathbb{N}\,. \end{aligned}$$

□

An Application of the Stopping Lemma

Example (Gambler's Ruin 2)
We consider the simple example of a martingale we encountered in the previous section – the random walk $(X_n)_{n\in\mathbb{N}}$. By the stopping lemma we conclude

$$\mathbb{E}X_n = \mathbb{E}X_1 = \mathbb{E}Z_1 = 0 \text{ for all } n \in \mathbb{N}.$$

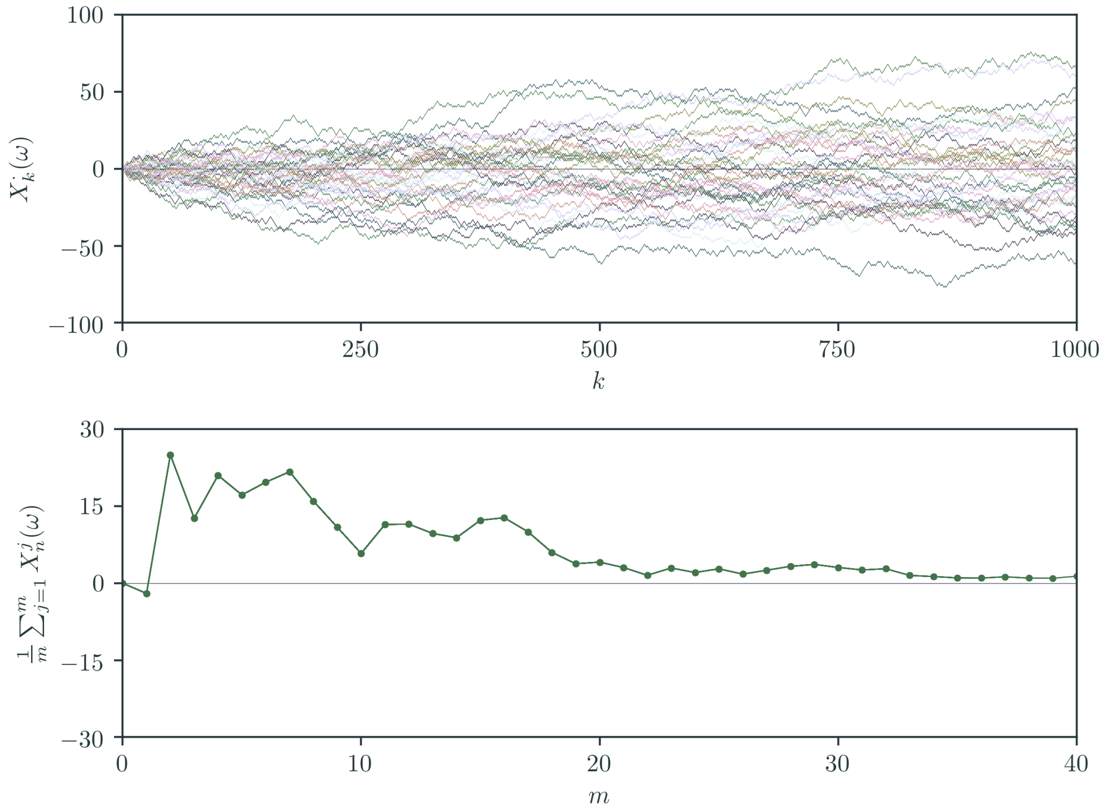

Figure 2: A realization of $s = 40$ independent copies $(X_k^j)_{k\le n}$ of $(X_k)_{k\le n}$ and their m-th sample mean at n, where $n = 10^3, m \le s, j \le s$.

Stopped Processes

Definition (Stopping Time)
Let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a filtration over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A random variable $T : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{N}_{\leq\infty}, 2^{\mathbb{N}_{\leq\infty}})$ such that

$$\{T \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}$$

is called a stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Remark
In the above setting,

$$T \text{ is a stopping time} \iff \{T = n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}.$$

Definition (Stopped Process)
Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to (E, \mathcal{E}), n \in \mathbb{N}$ be a process adapted to filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ over $(\Omega, \mathcal{A})$ and let $T : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{N}_{\leq\infty}, 2^{\mathbb{N}_{\leq\infty}})$ be a stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. We call

$$(X_n^T)_{n\in\mathbb{N}}, \text{ where } X_n^T(\omega) := X_{\min\{T(\omega), n\}}(\omega) \text{ for all } \omega \in \Omega, n \in \mathbb{N}$$

the stopped process.

Definition (σ-algebra of the T-past)
Let $T : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{N}_{\leq\infty}, 2^{\mathbb{N}_{\leq\infty}})$ be a stopping time with respect to filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ over $(\Omega, \mathcal{A})$. We call

$$\mathcal{A}_T := \{A \in \mathcal{A} \mid A \cap \{T \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$$

the σ-algebra of the T-past.

Theorem
Let $T : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{N}_{\leq\infty}, 2^{\leq\infty})$ be finite and a stopping time with respect to filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ over $(\Omega, \mathcal{A})$. Let further $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), n \in \mathbb{N}$ be a stochastic process adapted to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Then the random variable

$$X_T : \Omega \to \mathbb{R}\,,\ \omega \mapsto X_T(\omega) := X_{T(\omega)}(\omega)$$

is $\mathcal{A}_T$-$\mathcal{B}(\mathbb{R})$-measurable.

Proof.
Let $B \in \mathcal{B}(\mathbb{R})$ and $n \in \mathbb{N}$, then due to finiteness of T, we have

$$X_T^{-1}(B) = \bigcup_{j\in\mathbb{N}} X_j^{-1}(B) \cap \{T = j\} \in \mathcal{A}$$

and

$$X_T^{-1}(B) \cap \{T \leq n\} = \bigcup_{j=0}^{n} X_j^{-1}(B) \cap \{T = j\} \in \mathcal{F}_n \implies X_T^{-1}(B) \in \mathcal{A}_T\,.$$

□

Martingale Stopping Theorem

Theorem (Martingale Stopping Theorem)
If $(X_n)_{n\in\mathbb{N}}$ is a martingale with respect to filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and if T is a finite stopping time with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$, then

$$\mathbb{E}X_T = \mathbb{E}X_1$$

if one of the following holds:

- $(X_n)_{n\in\mathbb{N}}$ *is bounded* $\mathbb{P}$*-almost surely;*
- T *is bounded* $\mathbb{P}$*-almost surely; or*
- $\mathbb{E}T < \infty$, *and there is a constant* $c \in \mathbb{R}$ *such that*

$$\mathbb{E}(|X_{n+1} - X_n| \mid \mathcal{F}_n) < c \textit{ for all } n \in \mathbb{N}\,.$$

Proof Sketch.

- In all cases we have a finite T. Therefore, X_T is $\mathcal{A}_T$-$\mathcal{B}(\mathbb{R})$-measurable and in particular $\mathcal{A}$-$\mathcal{B}(\mathbb{R})$-measurable.
- There is a pointwise limit $X_n^T \to X_T$.
- By the requirements, one can uniformly bound the expected value $(\mathbb{E}X_n^T)_{n\in\mathbb{N}}$.
- Due to Lebesgue's Theorem we obtain $\lim_{n\to\infty} \mathbb{E}X_n^T = \mathbb{E}X_T$.
- By the Stopping Lemma, $\mathbb{E}X_n^T = \mathbb{E}X_1$ holds for all $n \in \mathbb{N}$.

$\square$

An Application of the Martingale Stopping Theorem

Example (Gambler's Ruin 3)
We extend the previous example toward a more interesting stopping time. We consider the same martingale - the random walk $(X_n)_{n\in\mathbb{N}}$ in conjunction with the stopping time

$$T^b(\omega) := \min\{n \in \mathbb{N} \mid X_n(\omega) \in \{\ell_1, -\ell_2\} \text{ or } n = b\} \text{ for all } \omega \in \Omega\,,$$

where $b \in \mathbb{N}_{\leq\infty}$ and $\ell_1, \ell_2 \in \mathbb{N}$.

- Clearly, T^b is a stopping time with respect to $\mathcal{F}_n$, as for $n \neq b$

$$\{T^b = n\} = \{X_n \in \{\ell_1, -\ell_2\}\} \cap \left(\bigcap_{k<n} \{X_k \notin \{\ell_1, -\ell_2\}\}\right) \in \mathcal{F}_n$$

  and in case $n = b < \infty$

$$\{T^b = b\} = \bigcap_{k<b} \{X_k \notin \{\ell_1, -\ell_2\}\} \in \mathcal{F}_b\,.$$

An Application of the Martingale Stopping Theorem

$T^b(\omega) := \min\{n \in \mathbb{N} \mid X_n(\omega) \in \{\ell_1, -\ell_2\} \text{ or } n = b\}$ for all $\omega \in \Omega$.

- We prove that $\mathbb{P}\{T^b = \infty\} = 0$. For this let $n \in \mathbb{N}$, define $\ell := \ell_1 + \ell_2$ and pick $r_n := \max\{k \in \mathbb{N} \mid k\ell \le n\}$. For all $b \in \mathbb{N}_{\le\infty}$ it follows, that

$$\begin{aligned}\{T^b \ge n\} \subseteq \{T^\infty \ge n\} &\subseteq \{T^\infty \ge r_n\ell\} \\ &\subseteq \bigcap_{k<r_n} \{|X_{(k+1)\ell} - X_{k\ell}| < \ell\} \\ &= \bigcap_{k<r_n} \{|X_{(k+1)\ell} - X_{k\ell}| = \ell\}^C \\ &= \bigcap_{0\le k<r_n} \left\{\left|\sum_{j=k\ell+1}^{(k+1)\ell} Z_j\right| = \ell\right\}^C\end{aligned}$$

and therefore

$$\mathbb{P}\{T^b \ge n\} \le (1 - 1/2^{\ell-1})^{r_n} \xrightarrow{n\to\infty} 0\,.$$

An Application of the Martingale Stopping Theorem

$T^b(\omega) := \min\{n \in \mathbb{N} \mid X_n(\omega) \in \{\ell_1, -\ell_2\} \text{ or } n = b\}$ for all $\omega \in \Omega$.

$\mathbb{P}\{T^b \geq n\} \xrightarrow{n\to\infty} 0$ for all $b \in \mathbb{N}_{\leq\infty}$.

- It remains to show that $\mathbb{E}X_{T^b} = 0$, where $b < \infty$. As T^b is bounded, we have

$$\mathbb{E}\,T^b \leq b < \infty$$

and due to the fact that

$$\mathbb{E}(|X_{n+1} - X_n| \mid \mathcal{F}_n) = \mathbb{E}(|Z_{n+1}| \mid \mathcal{F}_n) \leq \mathbb{E}(1 \mid \mathcal{F}_n) = 1\,,$$

we can apply the Martingale Stopping Theorem. Setting

$$q_b := \mathbb{P}(X_{T^b} = \ell_1) \quad \text{and} \quad w_b := \mathbb{P}(X_{T^b} \notin \{\ell_1, -\ell_2\})$$

we obtain

$$\ell_1 q_b - \ell_2(1 - q_b) \leq \mathbb{E}X_{T^b} = 0 \leq \ell_1(q_b + w_b) - \ell_2(1 - q_b - w_b)\,.$$

By the previous derivation, it follows additionally that

$$0 = \lim_{b\to\infty} \mathbb{E}X_{T^b} = \lim_{b\to\infty} \ell_1 q_b - \ell_2(1 - q_b) \iff \lim_{b\to\infty} q_b = \frac{\ell_2}{\ell_1 + \ell_2}\,.$$

An Application of the Martingale Stopping Theorem

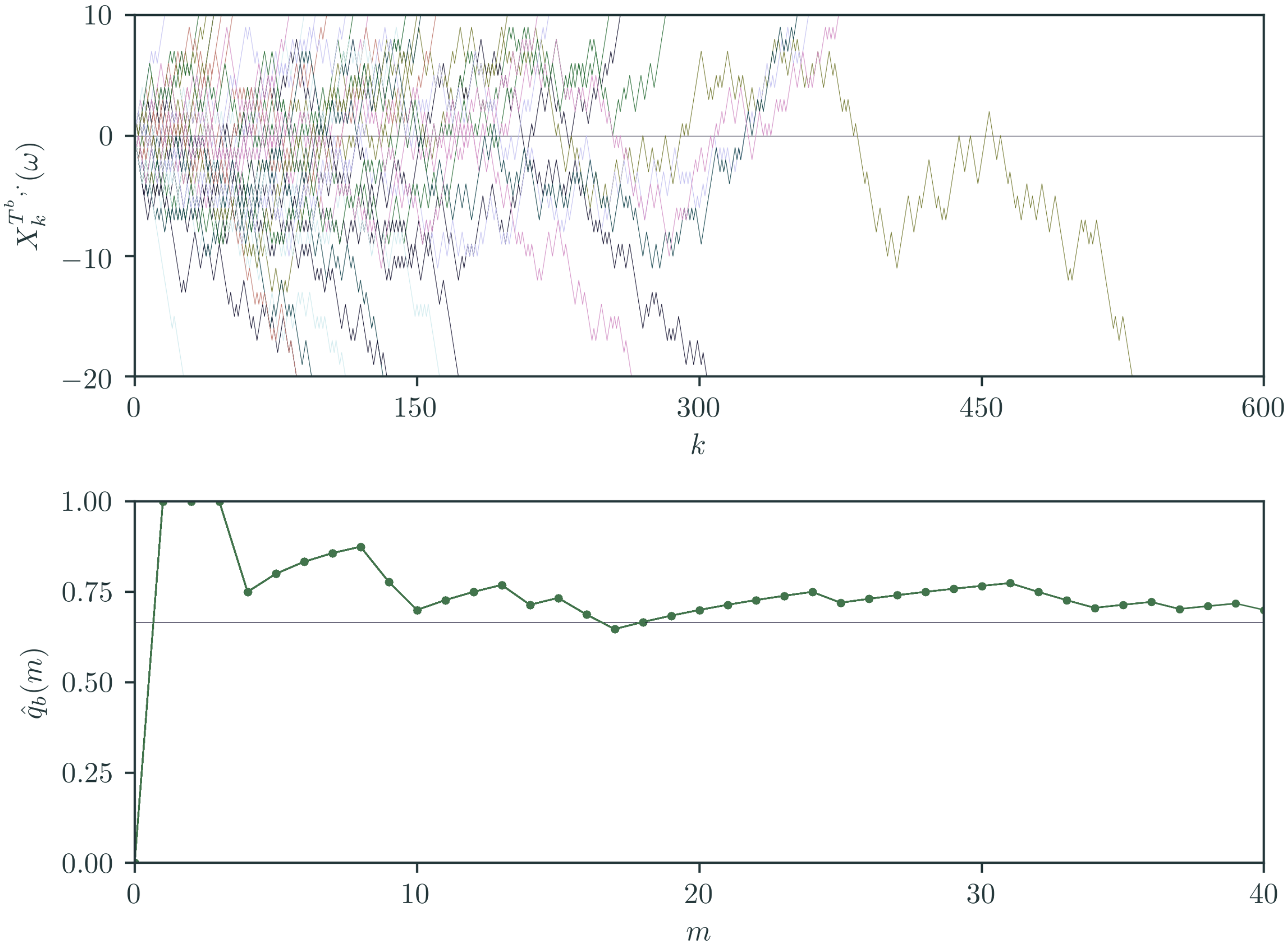

Figure 3: A realization of $s = 40$ independent copies $\left(X_k^{T_b,j}\right)_{k \leq n}$ of $\left(X_k^{T_b}\right)_{k \leq n}$ and the m-th sample mean $\hat{q}_b(m)$ of the corresponding realization of $\mathbb{1}\{X_T = \ell_1\}$, where $\ell_1 = 10; \ell_2 = 20; m, j \leq s; 0 \ll b$.

Wald's Equation

Wald's Equation

Theorem (Wald's Equation)
Let $X_n \in L^1(\Omega, \mathcal{A}, \mathbb{P}), n \in \mathbb{N}$ be independent, identically distributed random variables and let T be a finite stopping time with respect to

$$(\mathcal{F}_n := \mathcal{I}(X_1, X_2, \ldots, X_n))_{n\in\mathbb{N}} \,.$$

If T has bounded expectation, then

$$\mathbb{E}\left(\sum_{i=1}^{T} X_i\right) = \mathbb{E}\,T \cdot \mathbb{E}X_1 \,.$$

Proof.

- $(Z_n := \sum_{j=1}^{n}(X_j - \mathbb{E}X_j))_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.
- In conjunction with T , the martingale $(Z_n)_{n\in N}$ fulfills the third version of the Martingale Stopping Theorem.
- Due to linearity of the expected value and by the former, we obtain the result.

Wald's Equation

- $(Z_n := \sum_{j=1}^{n}(X_j - \mathbb{E}X_j))_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

Clearly, Z_n is $\mathcal{F}_n$-$\mathcal{B}(\mathbb{R})$-measurable and

$$\mathbb{E}|Z_n| \leq \mathbb{E}\left(\sum_{j=1}^{n}|X_j| + \mathbb{E}|X_j|\right) = 2n\mathbb{E}X_1 \text{ for all } n \in \mathbb{N}\,.$$

Also,

$$\begin{aligned}\mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) &= \mathbb{E}\left(X_{n+1} - \mathbb{E}X_{n+1} + Z_n \mid \mathcal{F}_n\right)\\ &= \mathbb{E}\left(X_{n+1} \mid \mathcal{F}_n\right) - \mathbb{E}X_{n+1} + Z_n\\ &= Z_n \text{ for all } n \in \mathbb{N}\,.\end{aligned}$$

Wald's Equation

- In conjuction with T, the martingale $(Z_n)_{n\in N}$ fulfills the third version of the Martingale Stopping Theorem.

We have

$$\begin{aligned}\mathbb{E}\left(|Z_{n+1}-Z_n| \mid \mathcal{F}_n\right) &= \mathbb{E}\left(|X_{n+1}-\mathbb{E}X_{n+1}| \mid \mathcal{F}_n\right)\\ &= \mathbb{E}\left(|X_{n+1}-\mathbb{E}X_{n+1}|\right)\\ &\leq 2\mathbb{E}|X_1| \text{ for all } n\in\mathbb{N}\end{aligned}$$

and as T is finite and $\mathbb{E}T<\infty$ by assumption, the Martingale Stopping Theorem tells us

$$\mathbb{E}Z_T=\mathbb{E}Z_1\,.$$

Wald's Equation

· Due to linearity of the expected value and by the former, we obtain the result.

Clearly,

$$\mathbb{E}Z_1 = \mathbb{E}(X_1 - \mathbb{E}X_1) = \mathbb{E}X_1 - \mathbb{E}X_1 = 0\,.$$

Therefore, we have

$$\begin{aligned} 0 = \mathbb{E}Z_1 = \mathbb{E}Z_T &= \mathbb{E}\left(\sum_{j=1}^{T}(X_j - \mathbb{E}X_1)\right) \\ &= \mathbb{E}\left(\left(\sum_{j=1}^{T} X_j\right) - T\mathbb{E}X_1\right) \\ &= \mathbb{E}\left(\sum_{j=1}^{T} X_j\right) - \mathbb{E}T \cdot \mathbb{E}X_1\,, \end{aligned}$$

which gives the result.

□

Concentration of Martingales

The Azuma-Hoeffding Inequality

Theorem (Azuma-Hoeffding Inequality)
Let $(X_n)_{n\in\mathbb{N}}$ be a martingale with respect to filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and assume that for all $k\in\mathbb{N}$, there exists $c_k\in\mathbb{R}$ such that $|X_{k+1}-X_k|\le c_k$.

Then, for all $n\in\mathbb{N}$ and for all $\lambda>0$, we have

$$\mathbb{P}\left(|X_n-X_1|\ge\lambda\right)\le 2\exp\left(-\frac{\lambda^2}{2\sum_{k=1}^{n-1}c_k^2}\right).$$

Proof.

- For every $\alpha>0$, there is an upper bound

$$\mathbb{E}\left(e^{\alpha(X_n-X_1)}\right)\le\exp\left(\frac{\alpha^2}{2}\sum_{k=1}^{n-1}c_k^2\right)\quad\text{for all } n\in\mathbb{N}.$$

- By Markov's Inequality and by subadditivity of $\mathbb{P}$, we therefore obtain a bound

$$\mathbb{P}(|X_n-X_1|\ge\lambda)\le 2\exp\left(\frac{\alpha^2}{2}\sum_{k=1}^{n-1}c_k^2-\alpha\lambda\right)\quad\text{for all } n\in\mathbb{N}.$$

- We minimize the upper bound over $\alpha>0$ to obtain the result.

The Azuma-Hoeffding Inequality

· For every $\alpha > 0$, there is an upper bound

$$\mathbb{E}\left(e^{\alpha(X_n - X_1)}\right) \leq \exp\left(\frac{\alpha^2}{2}\sum_{k=1}^{n-1} c_k^2\right) \text{ for all } n \in \mathbb{N}.$$

For all $n \in \mathbb{N}$, we define $Y_{n+1} := X_{n+1} - X_n$ and given $\alpha > 0$, we divide this step into the three substeps

- There is an upper bound

$$e^{\alpha Y_{n+1}} \leq \frac{e^{-\alpha c_n} + e^{\alpha c_n}}{2} + \frac{Y_{n+1}}{2 c_n}\left(e^{\alpha c_n} - e^{-\alpha c_n}\right);$$

- There is an upper bound

$$\mathbb{E}\left(e^{\alpha Y_{n+1}} \mid \mathcal{F}_n\right) \leq \exp\left({(\alpha c_n)^2}/{2}\right) \text{ ; and}$$

- There is an upper bound

$$\mathbb{E}\left(e^{\alpha(X_{n+1} - X_1)}\right) \leq \mathbb{E}\left(e^{\alpha(X_n - X_1)}\right) \exp\left({(\alpha c_n)^2}/{2}\right)$$

  from which the statement can be concluded by induction.

The Azuma-Hoeffding Inequality

◇ There is an upper bound

$$e^{\alpha Y_{n+1}} \leq \frac{e^{-\alpha c_n} + e^{\alpha c_n}}{2} + \frac{Y_{n+1}}{2c_n}\left(e^{\alpha c_n} - e^{-\alpha c_n}\right);$$

By assumption $|Y_{n+1}| = |X_{n+1} - X_n| \leq c_n$, therefore we can find a convex combination

$$Y_{n+1} = -c_n \frac{1 - {}^{Y_{n+1}}/_{c_n}}{2} + c_n \frac{1 + {}^{Y_{n+1}}/_{c_n}}{2}.$$

Using the convexity of $t \mapsto e^{\alpha t}$, we obtain that

$$\begin{aligned} e^{\alpha Y_{n+1}} &\leq \frac{1 - {}^{Y_{n+1}}/_{c_n}}{2} e^{-\alpha c_n} + \frac{1 + {}^{Y_{n+1}}/_{c_n}}{2} e^{\alpha c_n} \\ &= \frac{e^{-\alpha c_n} + e^{\alpha c_n}}{2} + \frac{Y_{n+1}}{2c_n}\left(e^{\alpha c_n} - e^{-\alpha c_n}\right). \end{aligned}$$

The Azuma-Hoeffding Inequality

◇ There is an upper bound

$$\mathbb{E}\left(e^{\alpha\, Y_{n+1}} \mid \mathcal{F}_n\right) \leq \exp\left({(\alpha\, c_n)^2}/{2}\right) \;;$$

Note that $(X_n)_{n\in\mathbb{N}}$ is a martingale with respect to filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$, thus

$$\begin{aligned}\mathbb{E}\,(\,Y_{n+1} \mid \mathcal{F}_n) &= \mathbb{E}\,(X_{n+1} - X_n \mid \mathcal{F}_n) \\ &= \mathbb{E}\,(X_{n+1} \mid \mathcal{F}_n) - X_n = 0\,.\end{aligned}$$

Using this and the previous result, we obtain

$$\begin{aligned}\mathbb{E}\left(e^{\alpha\, Y_{n+1}} \mid \mathcal{F}_n\right) &\leq \mathbb{E}\left(\frac{e^{\alpha\, c_n} + e^{-\alpha\, c_n}}{2} + \frac{Y_{n+1}}{2\, c_n}(e^{\alpha\, c_n} - e^{-\alpha\, c_n}) \mid \mathcal{F}_n\right) \\ &= \frac{e^{\alpha\, c_n} + e^{-\alpha\, c_n}}{2} = \tfrac{1}{2}\sum_{j=0}^{\infty} \frac{(\alpha\, c_n)^j + (-\alpha\, c_n)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{(\alpha\, c_n)^{2j}}{(2j)!} \leq \exp\left({(\alpha\, c_n)^2}/{2}\right)\,.\end{aligned}$$

The Azuma-Hoeffding Inequality

◇ There is an upper bound

$$\mathbb{E}\left(e^{\alpha(X_{n+1}-X_1)}\right) \leq \mathbb{E}\left(e^{\alpha(X_n-X_1)}\right) \exp\left((\alpha\, c_n)^2/2\right) \;;$$

From the previous statement and by the tower property it follows that

$$\begin{aligned}\mathbb{E}\left(e^{\alpha(X_{n+1}-X_1)}\right) &= \mathbb{E}\left(\mathbb{E}\left(e^{\alpha(X_n-X_1)}\, e^{\alpha\, Y_{n+1}} \mid \mathcal{F}_n\right)\right)\\ &= \mathbb{E}\left(e^{\alpha(X_n-X_1)}\mathbb{E}\left(e^{\alpha\, Y_{n+1}} \mid \mathcal{F}_n\right)\right)\\ &\leq \mathbb{E}\left(e^{\alpha(X_n-X_1)}\right) \exp\left((\alpha\, c_n)^2/2\right) \;.\end{aligned}$$

By induction and the fact that $e^{\alpha(X_1-X_1)} = 1$, we obtain

$$\mathbb{E}\left(e^{\alpha(X_n-X_1)}\right) \leq \exp\left(\frac{\alpha^2}{2}\sum_{k=1}^{n-1} c_k^2\right) \text{ for all } n \in \mathbb{N}\,.$$

The Azuma-Hoeffding Inequality

- By Markov's Inequality and by subadditivity of $\mathbb{P}$, we therefore obtain a bound

$$\mathbb{P}(|X_n - X_1| \geq \lambda) \leq 2\exp\left(\frac{\alpha^2}{2}\sum_{k=1}^{n-1} c_k^2 - \alpha\lambda\right) \quad \text{for all } n \in \mathbb{N}\,.$$

Using the previous result, we have for all $\alpha > 0$, for all $\lambda > 0$, and for all $n \in \mathbb{N}$ that

$$\begin{aligned} \mathbb{P}(X_n - X_1 \geq \lambda) &= \mathbb{P}\left(e^{\alpha(X_n - X_1)} \geq e^{\alpha\lambda}\right) \\ &\leq \frac{\mathbb{E}\left(e^{\alpha(X_n - X_1)}\right)}{e^{\alpha\lambda}} \\ &\leq \exp\left(\alpha^2 \sum_{k=1}^{n-1} c_k^2/2 - \alpha\lambda\right)\,. \end{aligned}$$

By the same argument applied to $(-X_n)_{n\in\mathbb{N}}$ we obtain the bound

$$\mathbb{P}(-X_n + X_1 \geq \lambda) \leq \exp\left(\alpha^2 \sum_{k=1}^{n-1} c_k^2/2 - \alpha\lambda\right)\,.$$

Subadditivity of $\mathbb{P}$ then yields the result.

The Azuma-Hoeffding Inequality

- We minimize the upper bound over $\alpha > 0$ to obtain the result.

For all $n \in \mathbb{N}$ and $\lambda > 0$, minimizing the upper bound of

$$\mathbb{P}(|X_n - X_1| \geq \lambda) \leq 2\exp\left(\frac{\alpha^2}{2}\sum_{k=1}^{n-1} c_k^2 - \alpha\lambda\right)$$

over $\alpha > 0$ is equivalent to minimizing the function

$$\alpha \longmapsto \frac{\alpha^2}{2}\sum_{k=1}^{n-1} c_k^2 - \alpha\lambda\,.$$

The minimum of the above function is found in its only local minimum and as the unique root

$$\alpha \sum_{k=1}^{n-1} c_k^2 - \lambda = 0 \iff \alpha = \frac{\lambda}{\sum_{k=1}^{n-1} c_k^2}\,.$$

of its derivative with respect to α. Plugging the minimum into the exponent of the above bound, we obtain

$$\frac{\alpha^2}{2}\sum_{k=1}^{n-1} c_k^2 - \alpha\lambda = \frac{1}{2}\frac{\lambda^2}{\sum_{k=1}^{n-1} c_k^2} - \frac{\lambda^2}{\sum_{k=1}^{n-1} c_k^2} = -\frac{\lambda^2}{2\sum_{k=1}^{n-1} c_k^2}$$

and conclude the result. □

An Application of the Azuma-Hoeffding Inequality

Definition (Graphs)
Define a dense graph $G_m := (V_m, E_m)$, where $m \in \mathbb{N}$,

- the set $V_m := \{v_j \mid j \in \mathbb{N},\ j \leq m\}$ is called vertices and
- the set
$$E_m := \{e : \{0,1\} \to V_m \mid e(0) \neq e(1)\} /_\sim ,$$
where
$$e \sim e' :\iff e\{0,1\} = e'\{0,1\},$$
is called edges.

We say $G := (V, E)$ is a graph k-subordinate to G_m if

- $V = V_m$,
- $E \subseteq E_m$, and
- $|E| = k$.

We write $\mathcal{G}_m^k := \{G \mid G \text{ is } k\text{-subordinate to } G_m\})$. Further, we call a vertice $v \in V$ isolated if
$$v \notin \bigcup_{e \in E} e\{0,1\}.$$

An Application of the Azuma-Hoeffding Inequality

Example (Concentration of the Number of Isolated Vertices)
We will now take a look at random variable

$$G \sim \mathrm{Unif}(\mathcal{G}_m^k)\,.$$

Clearly, as the range of G is finite, we can consider the associated power set as a σ-algebra. Of special interest to us are the number of isolated vertices $\mathrm{iso}(G)$, constructed with the map

$$\begin{cases} \mathrm{iso} : \mathcal{G}_m^k \longrightarrow \mathbb{N}_{[0,m]} \\ (V, E) \longmapsto |\{v \in V \mid v \text{ is isolated}\}|\,. \end{cases}$$

We will now

- Compute $\mathbb{E}\,\mathrm{iso}(G)$ and
- Prove that $\mathbb{P}\{|\mathrm{iso}(G) - \mathbb{E}\,\mathrm{iso}(G)| \geq \lambda\} \leq 2\exp\left(-\frac{\lambda^2}{8k}\right)$.

An Application of the Azuma-Hoeffding Inequality

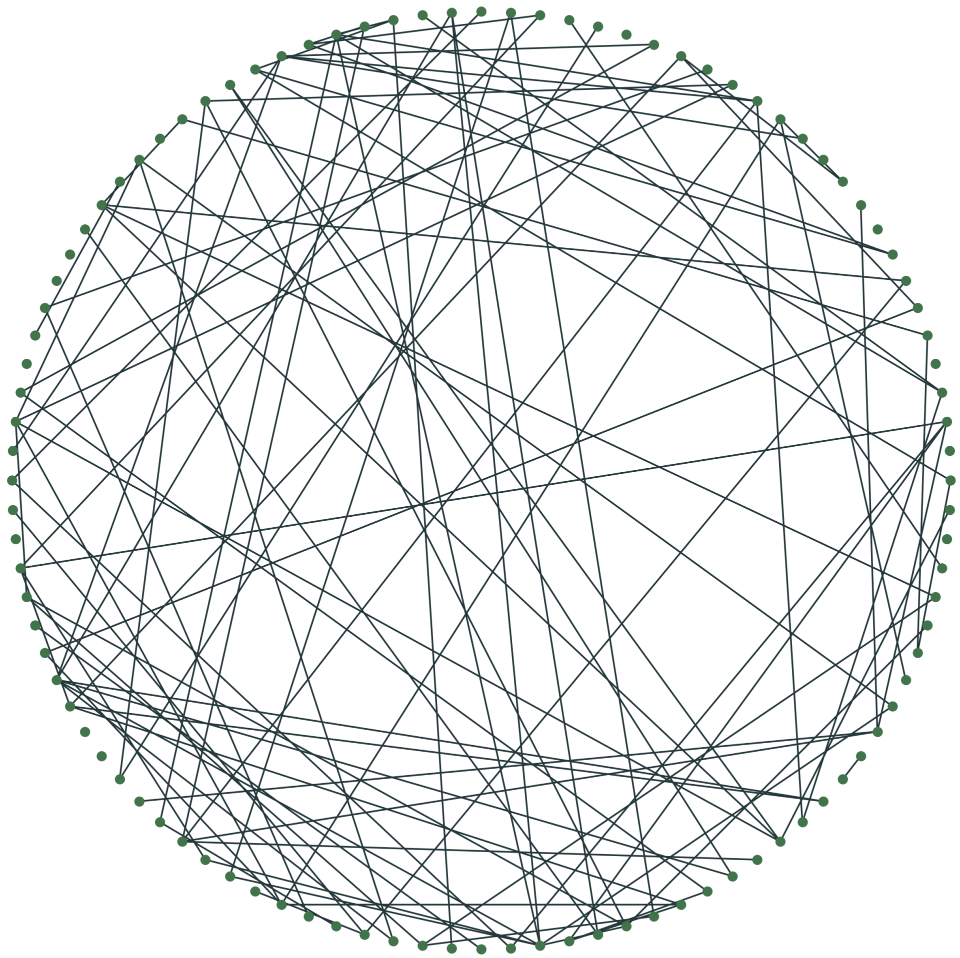

Figure 4: A realization of $G \sim \mathrm{Unif}(\mathcal{G}_m^k)$, where $m = k = 100$.

An Application of the Azuma-Hoeffding Inequality

- Compute $\mathbb{E}\,\mathrm{iso}(G)$;

We have the identity

$$\begin{aligned}\mathbb{E}\,\mathrm{iso}(G) &= \mathbb{E}\sum_{v\in V} \mathbb{1}\{v \text{ is isolated}\} \\ &= \sum_{v\in V} \mathbb{E}\,\mathbb{1}\{v \text{ is isolated}\} \\ &= |V| \cdot \mathbb{P}\{v_1 \text{ is isolated}\}\,.\end{aligned}$$

Clearly, $|V| = |V_m| = m$. As the random variable G takes exactly the values of every graph containing k edges, we have

$$|\mathcal{G}_m^k| = \binom{\binom{m}{2}}{k}\,.$$

For the probability of a vertice being isolated we count the number of graphs that fulfill this property and obtain the associated probability by division

$$\mathbb{P}\{v_1 \text{ is isolated}\} = \frac{|\mathcal{G}_{m-1}^k|}{|\mathcal{G}_m^k|}\,.$$

An Application of the Azuma-Hoeffding Inequality

· Prove that $\mathbb{P}\{|\mathrm{iso}(G) - \mathbb{E}\,\mathrm{iso}(G)| \geq \lambda\} \leq 2\exp\left(-\frac{\lambda^2}{8k}\right)$.

This follows, as we can

- ◇ Build an edge exposure martingale $(X_n)_{n\in\mathbb{N}}$ of $\mathrm{iso}(G)$ with respect to G;
- ◇ Bound $|X_{n+1} - X_n| \leq 2$ for all $n \in \mathbb{N}$; and
- ◇ Show that the result can be obtained by applying the Azuma-Hoeffding Inequality to the martingale $(X_n)_{n\in\mathbb{N}}$.

An Application of the Azuma-Hoeffding Inequality

◇ Build an edge exposure martingale $(X_n)_{n\in\mathbb{N}}$ of $\mathrm{iso}(G)$ with respect to G;

Consider an enumeration $(e)_{i\leq j}$, $j = \frac{m(m-1)}{2}$ of the edges E_m of a dense graph with m vertices. We will now define a random variable describing the vertices that are connected to e_i in the random graph G. For $1 \leq i \leq j$, we define

$$\begin{cases} \tilde{e}_i : \mathcal{G}_n^k \longrightarrow 2^V \\ (V, E) \longmapsto e_i\{0,1\} \text{ if } e_i \in E\,,\ \emptyset \text{ else}\,. \end{cases}$$

and therefore $\tilde{e}_i(G)$ describes just that.
Now for all $n \in \mathbb{N}$, we define $\mathcal{F}_n := \mathcal{I}(\tilde{e}_1(G), \ldots, \tilde{e}_n(G))$ and

$$X_0 := \mathbb{E}\,\mathrm{iso}(G)\,,\ X_n := \mathbb{E}(\mathrm{iso}(G) \mid \mathcal{F}_n)\,.$$

Clearly, $(X_n)_{n\in\mathbb{N}_{\geq 0}}$ is a martingale as for all $n \in \mathbb{N}_{\geq 0}$, we have

$$\begin{aligned} \mathbb{E}(X_{n+1} \mid \mathcal{F}_n) &= \mathbb{E}(\mathbb{E}(\mathrm{iso}(G) \mid \mathcal{F}_{n+1}) \mid \mathcal{F}_n) \\ &= \mathbb{E}(\mathrm{iso}(G) \mid \mathcal{F}_n) = X_n\,. \end{aligned}$$

An Application of the Azuma-Hoeffding Inequality

◇ Bound $|X_{n+1} - X_n| \leq 2$ for all $n \in \mathbb{N}$;

By definition, we have for $G = (V, E) \sim \mathrm{Unif}(\mathcal{G}_m^k)$, that

$$\mathrm{iso}(G) = \left|\left(\bigcup_{e \in E} e\{0,1\}\right)^C\right| = \left|\left(\bigcup_{i \leq j} \tilde{e}_i(G)\right)^C\right| \leq \left|\left(\bigcup_{\substack{i \leq j \\ i \neq n+1}} \tilde{e}_i(G)\right)^C\right|.$$

Therefore, for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} X_{n+1} - X_n &= \mathbb{E}(\mathrm{iso}(G) \mid \mathcal{F}_{n+1}) - \mathbb{E}(\mathrm{iso}(G) \mid \mathcal{F}_n) \\ &\leq \mathbb{E}\left(\left|\bigcap_{\substack{i \leq j \\ i \neq n+1}} \tilde{e}_i(G)^C\right| \,\middle|\, \mathcal{F}_{n+1}\right) - \mathbb{E}(\mathrm{iso}(G) \mid \mathcal{F}_n) \\ &= \mathbb{E}\left(\left|\bigcap_{\substack{i \leq j \\ i \neq n+1}} \tilde{e}_i(G)^C\right| - \mathrm{iso}(G) \,\middle|\, \mathcal{F}_n\right) \leq \mathbb{E}\left(|\tilde{e}_{n+1}(G)| \mid \mathcal{F}_n\right) \\ &\leq 2\,. \end{aligned}$$

By a similar derivation we get $X_{n+1} - X_n \geq -2$ and conclude the result.

An Application of the Azuma-Hoeffding Inequality

◇ Show that the result can be obtained by applying the Azuma-Hoeffding Inequality to the martingale $(X_n)_{n\in\mathbb{N}}$.

Clearly, $X_n = \mathrm{iso}(G)$ if $n \geq \frac{m(m-1)}{2}$ and $X_0 = \mathbb{E}\,\mathrm{iso}(G)$. By the fact that G has only k edges and by the Azuma-Hoeffding Inequality, we get

$$\mathbb{P}\{|\mathrm{iso}(G) - \mathbb{E}\,\mathrm{iso}(G)| \geq \lambda\} \leq 2\exp\left(-\frac{\lambda^2}{2\sum_{i=1}^{m(m-1)/2} c_i^2}\right) = 2\exp\left(-\frac{\lambda^2}{8k}\right) .$$

An Application of the Azuma-Hoeffding Inequality

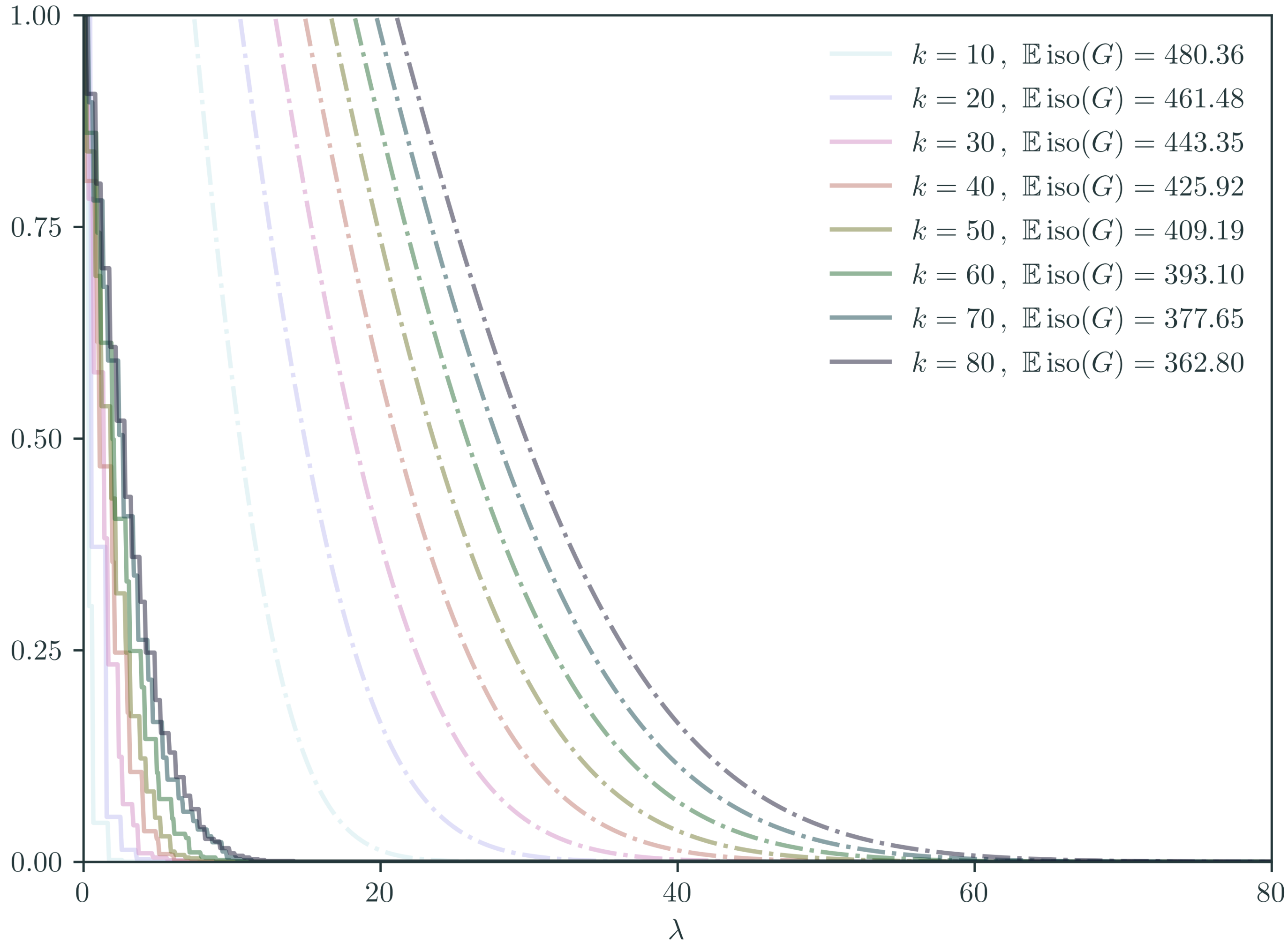

Figure 5: An estimation of $\lambda \mapsto \mathbb{P}\{|\mathrm{iso}(G) - \mathbb{E}\,\mathrm{iso}(G)| \geq \lambda\}$, created from 1000 independent samples of $\mathrm{Unif}(\mathcal{G}_m^k)$, and the associated bounds (dash-dotted) for $m = 500$ and different values of k .

Thank You!

Find the full source-code of this presentation on my Github soon:

github.com/NiMlr

The content of this presentation is mainly based on and influenced by:

- M. Mitzenmacher and E. Upfal (2017). *Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis.* Cambridge University Press. URL: https://books.google.de/books?id=E9UlDwAAQBAJ
- N. Henze (2016). *Vorlesung Wahrscheinlichkeitstheorie.* KIT Bibliothek. URL: https://publikationen.bibliothek.kit.edu/1000113898

The source-code to generate the visualizations was written in *Python 3* and makes use of the software:

- P. Virtanen et al. (2020). "SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python". In: *Nature Methods* 17, pp. 261–272
- F. Pedregosa et al. (2011). "Scikit-learn: Machine Learning in Python". In: *Journal of Machine Learning Research* 12, pp. 2825–2830
- A. Hagberg, D. Schult, and P. Swart (2008). "Exploring Network Structure, Dynamics, and Function using NetworkX". In: *Proceedings of the 7th Python in Science Conference.* Pasadena, CA USA, pp. 11–15