

Probability Theory I

Notes of a lecture held in winter 2019/2020
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In the accompanying lecture, we developed the mathematical foundations of probability theory. The topics of this course form the basis for all further lectures in probability and statistics held at Ruhr-Universität Bochum. Prerequisites are the Introduction to Probability and Statistics, Linear Algebra and Geometry I/II as well Analysis I-III. Since a particular focus has been on analytic methods, some basic knowledge in complex analysis is helpful but not strictly required. Prior experience with measure theory is also useful.

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1 A Crash Course in Measure Theory

1.1 Elementary Definitions in Measure Theory

Definition 1.1 (σ -algebra, Measurable space)

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Let $\Omega \neq \emptyset$. $\mathcal{A} \subseteq 2^\Omega$ is called a σ -algebra over Ω if

1. $\Omega \in \mathcal{A}$,
2. $A \in \mathcal{A} \Rightarrow A^C \in \mathcal{A}$ and
3. $A_1, A_2, \dots \in \mathcal{A} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$.

(Ω, \mathcal{A}) is called a measurable space.

Definition 1.2 (Measure, Measure space)

Let (Ω, \mathcal{A}) be a measurable space, then a function $\mu : \mathcal{A} \rightarrow [0, \infty]$ is called a measure on (Ω, \mathcal{A}) if

- $\mu(\emptyset) = 0$ and
- μ is σ -additive, i.e. for all $A_1, A_2, \dots \in \mathcal{A}$ pairwise disjoint, we have $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$.

We call $(\Omega, \mathcal{A}, \mu)$ a measure space.

Definition 1.3 (Probability measure, Probability space, Event)

If μ is a measure on (Ω, \mathcal{A}) with $\mu(\Omega) = 1$, we call it a probability measure, $(\Omega, \mathcal{A}, \mu)$ a probability space and $A \in \mathcal{A}$ an event.

Remark 1.1 We write $\mathbb{P} := \mu$ if μ is a probability measure.

Further Concepts Related to Measures

· *Generated σ -algebra:* Let $\mathcal{M} \subseteq 2^\Omega$, then

$$\sigma(\mathcal{M}) := \bigcap_{\substack{\mathcal{A} \text{ } \sigma\text{-algebra} \\ \mathcal{M} \subseteq \mathcal{A}}} \mathcal{A}$$

is called the generated σ -algebra of \mathcal{M} over Ω .

· *Initial σ -algebra:* Let I be an arbitrary index set and for $i \in I$ let $f_i : \Omega \rightarrow \Omega_i$ be a function and $(\Omega_i, \mathcal{A}_i)$ be a measurable space. We call

$$\mathcal{I}((f_i)_{i \in I}) := \sigma\left(\bigcup_{i \in I} f_i^{-1}(\mathcal{A}_i)\right)$$

the initial σ -algebra on Ω generated by $(f_i)_{i \in I}$.

· *Borel σ -algebra:* Let (T, τ) be a topological space, then then then Borel σ -algebra of T over τ is defined as $\mathcal{B}(T) := \sigma(\tau)$.

· *Finite measure:* Measure μ on Ω is finite if $\mu(\Omega) < \infty$.

· *σ -finite measure:* Ω can be covered by at most countably measurable sets in \mathcal{A} with finite measure with respect to μ .

· *Dirac measure:* Let $\omega \in \Omega$, \mathcal{A} a σ -algebra over Ω and $A \in \mathcal{A}$, then

$$\delta_\omega(A) := \mathbb{1}_A(\omega) := \begin{cases} 0, & \omega \notin A; \\ 1, & \omega \in A \end{cases}$$

is called a Dirac measure on Ω .

· *Semiring:* $\mathcal{H} \subseteq 2^\Omega$ is called a semiring iff

1. $\emptyset \in \mathcal{H}$,
2. \mathcal{H} is \cap -stable, i.e. $A, B \in \mathcal{H} \Rightarrow A \cap B \in \mathcal{H}$ and
3. $A, B \in \mathcal{H} \Rightarrow \exists C_1, \dots, C_n \in \mathcal{H}$ pairwise disjoint with $A \setminus B = C_1 \cup \dots \cup C_n$.

Example: $\Omega = \mathbb{R}, \mathcal{H} = \{(a, b] \mid a \leq b\}$.

· *Content:* $\mu : \mathcal{H} \rightarrow [0, \infty]$ such that $\mu(\emptyset) = 0$, μ finitely additive is called a content.

· *Pre-measure:* $\mu : \mathcal{H} \rightarrow [0, \infty]$ such that μ is a content and μ is σ -additive is called a pre-measure.

1.2 Important Theorems in Measure Theory

Theorem 1.1 (Uniqueness theorem for measures)

Let (Ω, \mathcal{A}) be a measure space, $\mathcal{M} \subseteq 2^\Omega$ \cap -stable with $\sigma(\mathcal{M}) = \mathcal{A}$ and μ_1, μ_2 measures on (Ω, \mathcal{A}) with

$$\mu_1(B) = \mu_2(B) \text{ for all } B \in \mathcal{M}.$$

If there is a sequence $(B_n)_{n \in \mathbb{N}}$ in \mathcal{M} with $B_n \uparrow \Omega$ and $\mu_1(B_n) < \infty$ for all $n \in \mathbb{N}$, then $\mu_1 = \mu_2$.

Theorem 1.2 (Extension theorem for measures)

Let $\mathcal{H} \subseteq 2^\Omega$ be a semiring and $\mu : \mathcal{H} \rightarrow [0, \infty]$ be a pre-measure, then there is a measure $\tilde{\mu}$ on $(\Omega, \sigma(\mathcal{H}))$ with

$$\tilde{\mu}(A) = \mu(A) \text{ for all } A \in \mathcal{H}.$$

If μ is σ -finite, then $\tilde{\mu}$ is unique.

Definition 1.4 (Measure determining function)

A function $G : \mathbb{R} \rightarrow \mathbb{R}$ is called measure determining if

1. G is increasing and
2. G is right continuous.

If $G(-\infty) = 0, G(\infty) = 1$, then G is called a distribution function.

Theorem 1.3 Let G be a measure determining function, then there exists a unique measure μ_G on $\mathcal{B}(\mathbb{R})$ with

$$\mu([a, b]) = G(b) - G(a), \text{ for all } a, b \in \mathbb{R} \text{ with } a \leq b.$$

If G is a distribution function, μ is a probability measure.

Example 1.1 Let $G(x) = x$, then μ_G is called the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition 1.5 (Measurable function)

Let $(\Omega, \mathcal{A}), (\Omega', \mathcal{A}')$ be measurable spaces. A function $f : \Omega \rightarrow \Omega'$ is called \mathcal{A} - \mathcal{A}' -measurable if

$$f^{-1}(A') \in \mathcal{A}, \text{ for all } A' \in \mathcal{A}'.$$

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Remark 1.2 If the respective measure spaces referred to are clear by context, we also say f is measurable.

Theorem 1.4 Let μ be a measure on Ω and $f : \Omega \rightarrow \Omega'$ be \mathcal{A} - \mathcal{A}' -measurable, then

$$\mu^f(A') := \mu(f^{-1}(A'))$$

determines a measure on (Ω', \mathcal{A}') . μ^f is called image measure of μ under f .

Remark 1.3 We write μ^f or $\mu \circ f^{-1}$ for the image measure of μ under f .

Definition 1.6 (Random variable, Distribution)

If $\mu = \mathbb{P}$ is a probability measure on (Ω, \mathcal{A}) , then a measurable function $X : (\Omega, \mathcal{A}, \mu) \rightarrow (\Omega', \mathcal{A}')$ is called a random variable with values in Ω' . We call $\mathbb{P}^X = \mathbb{P} \circ X^{-1}$ the distribution of X .

Remark 1.4 To denote that ν is the distribution of a random variable X , we also write $X \sim \nu$. If X, Y are random variables with equal distributions, then we write $X \stackrel{d}{=} Y$ or $X \sim Y$.

Important Facts about Measurable Functions

- Let $(\Omega, \mathcal{A}), (\Omega', \mathcal{A}')$ be measurable spaces and $f : \Omega \rightarrow \Omega'$. If

$$\mathcal{A}' = \sigma(\mathcal{M}') \text{ and } f^{-1}(\mathcal{M}') \subseteq \mathcal{A},$$

then f is \mathcal{A} - \mathcal{A}' -measurable.

- If additionally \mathcal{A}' is generated by a topology, one speaks about Borel measurability.
- If $\Omega' = \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ we have to consider the σ -algebra

$$\mathcal{B}(\overline{\mathbb{R}}) = \{B \cup E \mid B \in \mathcal{B}(\mathbb{R}), E \subseteq \{\pm\infty\}\}.$$

1.3 Construction of an Integral for Measurable Functions

Definition 1.7 (Integral)

The integral is constructed in four steps. Let in the following $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be \mathcal{A} - $\mathcal{B}(\overline{\mathbb{R}})$ -measurable.

1. If $f = \mathbb{1}_A$ for $A \in \mathcal{A}$, then

$$\int_{\Omega} f \, d\mu = \int_{\Omega} \mathbb{1}_A \, d\mu := \mu(A).$$

2. If instead f is a non-negative step function, i.e.

$$f \in \mathcal{E} := \{g : \Omega \rightarrow \overline{\mathbb{R}} \mid g \geq 0 \text{ is } \mathcal{A}\text{-}\mathcal{B}(\overline{\mathbb{R}})\text{-measurable, } |g(\Omega)| < \infty\},$$

then there exist $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{R}}_{\geq 0}$ and $A_1, \dots, A_n \in \mathcal{A}$ such that

$$f = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}.$$

In this case, we can define the integral of f with respect to μ as

$$\int_{\Omega} f \, d\mu := \sum_{i=1}^n \alpha_i \int_{\Omega} \mathbb{1}_{A_i} \, d\mu = \sum_{i=1}^n \alpha_i \mu(A_i).$$

3. If instead f is a non-negative function, i.e. $f(\Omega) \subseteq [0, \infty]$, then as shown by Lebesgue there exists $(u_n)_{n \in \mathbb{N}} \in \mathcal{E}$, such that $u_n \uparrow f$ pointwise convergent and we define

$$\int_{\Omega} f \, d\mu := \lim_{n \rightarrow \infty} \int_{\Omega} u_n \, d\mu.$$

It can be shown that this definition is independent of the choice of sequence $(u_n)_{n \in \mathbb{N}}$.

4. Otherwise decompose $f = f_{\geq 0} - f_{\leq 0}$, where

$$f_{\geq 0}(\omega) := \max\{f(\omega), 0\} \quad \text{and} \quad f_{\leq 0} := -\min\{f(\omega), 0\}.$$

We call f μ -integrable if

$$\int_{\Omega} f_{\geq 0} \, d\mu < \infty \quad \text{and} \quad \int_{\Omega} f_{\leq 0} \, d\mu < \infty.$$

In that case, we define

$$\int_{\Omega} f \, d\mu := \int_{\Omega} f_{\geq 0} \, d\mu - \int_{\Omega} f_{\leq 0} \, d\mu.$$

Remark 1.5 If the respective space referred to is clear by context, we also write $\int f \, d\mu := \int_{\Omega} f \, d\mu$ and call it the μ -integral of f .

Corollary 1.4.1 (Properties of the integral)

Let $f, g : \Omega \rightarrow \overline{\mathbb{R}}$ be μ -integrable, then

1. for all $a, b \in \mathbb{R}$, we have $\int_{\Omega} af + bg \, d\mu = a \int_{\Omega} f \, d\mu + b \int_{\Omega} g \, d\mu$,
2. from $f \leq g$ follows $\int_{\Omega} f \, d\mu \leq \int_{\Omega} g \, d\mu$ and
3. $|\int_{\Omega} f \, d\mu| \leq \int_{\Omega} |f| \, d\mu$.

Definition 1.8 (Almost everywhere, Almost surely)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let E_{ω} be a proposition for every $\omega \in \Omega$. We say E holds μ -almost everywhere (μ -a.e.) if there exists $N \in \mathcal{A}$ with $\mu(N) = 0$ and for all $\omega \in \Omega \setminus N$ we have E_{ω} . If $\mu = \mathbb{P}$ is a probability measure, we say that E holds \mathbb{P} -almost surely (\mathbb{P} -a.s.).

Definition 1.9 (L^p space)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $p \in [1, \infty)$. The space $L^p(\Omega, \mathcal{A}, \mu)$ consists of equivalence classes of measurable functions $f : (\Omega, \mathcal{A}, \mu) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with

$$\int_{\Omega} |f|^p \, d\mu < \infty.$$

In this space, two measurable functions with the above properties are considered equivalent if they are equal μ -almost everywhere.

Remark 1.6 With respect to the previous definition, we define a norm on $L^p(\Omega, \mathcal{A}, \mu)$ as

$$\|f\|_p := \left(\int_{\Omega} |f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Assuming instead that for $f : (\Omega, \mathcal{A}, \mu) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ measurable,

$$\|f\|_{\infty} := \sup \left\{ M \in [0, \infty) \mid \mu(\{\omega \in \Omega \mid |f(\omega)| > M\}) = 0 \right\}$$

exists. We denote the set consisting of equivalence classes of functions fulfilling the previous property which are equal μ -almost everywhere with $L^{\infty}(\Omega, \mathcal{A}, \mu)$.

Definition 1.10 (Expectation of a random variable)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $X : \Omega \rightarrow \mathbb{R}$ a random variable. We say the expectation of X exists if X is \mathbb{P} -integrable. In that case, we define the expectation of X as

$$\mathbb{E}(X) := \int_{\Omega} X \, d\mathbb{P}.$$

1.4 Important Integral Theorems

Theorem 1.5 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f, g : \Omega \rightarrow \overline{\mathbb{R}}$ be measurable with $f = g$ μ -almost everywhere, then ▷ 15.10.2019

$$\int_{\Omega} f \, d\mu = \int_{\Omega} g \, d\mu.$$

Theorem 1.6 (Markov's inequality)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $f : \Omega \rightarrow \mathbb{R}_{\geq 0}$ be μ -integrable and $t > 0$, then

$$\mu(f^{-1}[t, \infty)) \leq \frac{1}{t} \int_{\Omega} f \, d\mu.$$

Corollary 1.6.1 Let $f : \Omega \rightarrow \overline{\mathbb{R}}$ be μ - $\mathcal{B}(\overline{\mathbb{R}})$ -measurable.

1. If $f \geq 0$, then $\int f \, d\mu = 0 \Leftrightarrow f = 0$ μ -almost everywhere.
2. If $\int |f| \, d\mu < \infty$, then $|f| < \infty$ μ -almost everywhere.

Remark 1.7 The former can be formulated in probabilistic terms. If X is a random variable with values in $\overline{\mathbb{R}}$, the following holds.

1. If $X \geq 0$, then $\mathbb{E}X = 0 \Leftrightarrow X = 0$ \mathbb{P} -almost surely.
2. If $\mathbb{E}(|X|) < \infty$, then $|X| < \infty$ \mathbb{P} -almost surely.

Theorem 1.7 (Monotone convergence)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f_1, f_2, \dots : \Omega \rightarrow [0, \infty]$ be a sequence of measurable functions. If $(f_n)_{n \in \mathbb{N}}$ is also pointwise non-decreasing μ -almost everywhere, then the integral exists and

$$\int \lim_{n \rightarrow \infty} f_n \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu.$$

Theorem 1.8 (Fatou's lemma)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f_1, f_2, \dots : \Omega \rightarrow [0, \infty]$ be a sequence of measurable functions, then

$$\int \liminf_{n \rightarrow \infty} f_n \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu.$$

Theorem 1.9 (Lebesgue dominated convergence theorem)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $f_1, f_2, \dots : \Omega \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions with pointwise $\lim_{n \rightarrow \infty} f_n = f$ μ -almost everywhere. If there exists a μ -integrable function $g : \Omega \rightarrow [0, \infty]$ with

$$|f_n| \leq g, \quad \mu\text{-almost everywhere, for all } n \in \mathbb{N},$$

then f is μ -integrable and

$$\int f \, d\mu = \lim_{n \rightarrow \infty} \int f_n \, d\mu.$$

The Principle of Measure Theoretic Induction

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. If the goal is to show that an integral property E_f holds for all measurable functions $f : \Omega \rightarrow \overline{\mathbb{R}}$, one may proceed as follows.

1. Prove that E_f holds for non-negative step functions, i.e. $f \in \mathcal{E}$ and in particular for indicators $\mathbb{1}_A, A \in \mathcal{A}$.
2. Use the Lebesgue monotone convergence theorem to show that E_f holds for all non-negative measurable functions f .
3. Show that the property E_f holds for measurable functions f , by decomposing $f = f_{\geq 0} - f_{\leq 0}$.

Theorem 1.10 (Integration w.r.t. image measures)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, (Ω', \mathcal{A}') a measurable space, $f : \Omega \rightarrow \Omega'$ and $h : \Omega' \rightarrow \overline{\mathbb{R}}$ both measurable. We observe the following properties.

1. If $h \geq 0$, then

$$\int_{\Omega'} h \, d\mu^f = \int_{\Omega} h \circ f \, d\mu.$$

2. h is μ^f -integrable $\Leftrightarrow h \circ f$ is μ -integrable. The formula above holds as well.

1.5 Measures with Density

Definition 1.11 (Integration over subsets)

Let $f : \Omega \rightarrow \overline{\mathbb{R}}$ be \mathcal{A} - $\mathcal{B}(\overline{\mathbb{R}})$ -measurable and f non-negative or μ -integrable. For $A \in \mathcal{A}$, we define

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$$\int_A f \, d\mu := \int_{\Omega} f \cdot \mathbb{1}_A \, d\mu$$

as the μ -integral of f over A .

Remark 1.8 One needs to show that the former is well-defined, by showing, that $f \cdot \mathbb{1}_A$ is μ -integrable.

Theorem 1.11 In the above situation if $f \geq 0$ μ -almost everywhere,

$$A \in \mathcal{A} \mapsto f\mu(A) := \int_A f \, d\mu$$

defines a measure on (Ω, \mathcal{A}) .

Definition 1.12 (Measures with density)

We call $f\mu$ the measure with density f with respect to μ .

Remark 1.9 If $\Omega = \mathbb{R}^k$ for some $k \in \mathbb{N}$ and μ is the Lebesgue measure, f is called Lebesgue density.

Theorem 1.12 (Uniqueness of densities)

Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, $f, g : \Omega \rightarrow \overline{\mathbb{R}}$ measurable and $f, g \geq 0$.

1. If $f = g$ μ -almost everywhere, then $f\mu = g\mu$.
2. If f or g is μ -integrable, then the former holds in both directions.

Theorem 1.13 Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and $f \mu$ a measure with density.

1. If $\varphi : \Omega \rightarrow [0, \infty]$ measurable, then

$$\int_{\Omega} \varphi \, d f \mu = \int_{\Omega} \varphi \cdot f \, d \mu.$$

2. If $\varphi : \Omega \rightarrow \overline{\mathbb{R}}$ is measurable, then

$$\varphi \text{ is } f \mu\text{-integrable} \iff \varphi \cdot f \text{ is } \mu\text{-integrable}.$$

If the latter holds, then the statement in 1. holds as well.

Definition 1.13 (Absolute continuity of measures)

Let (Ω, \mathcal{A}) be a measurable space and μ, ν be associated measures. We say ν is absolutely continuous with respect to μ if for all $A \in \mathcal{A}$

$$\mu(A) = 0 \implies \nu(A) = 0$$

and we write $\nu \ll \mu$.

Theorem 1.14 (Radon-Nikodym)

Let (Ω, \mathcal{A}) be a measurable space and μ, ν be associated measures. If μ is σ -finite, then

$$\nu \ll \mu \iff \nu \text{ has a density with respect to } \mu.$$

Remark 1.10

- σ -finiteness of μ is necessary for the existence of a density with respect to ν .
- The density is μ -almost everywhere uniquely determined. We denote it by $\frac{d\nu}{d\mu}$.
- In the previous lecture we called a random variable continuous if $\mathbb{P}^X \ll \lambda$.

Definition 1.14 (Singular measures)

Let (Ω, \mathcal{A}) be a measurable space and μ, ν be associated measures.

We say μ and ν are singular if there exists $A \in \mathcal{A}$ such that $\mu(A) = 0$ and $\nu(\Omega \setminus A) = 0$. In this case we write $\mu \perp \nu$.

Theorem 1.15 (Lebesgue decomposition)

Let μ and ν be measures on (Ω, \mathcal{A}) with ν being σ -finite. Then there are uniquely determined measures ν_a and ν_s with

$$\nu_a \ll \mu, \nu_s \perp \mu \text{ and } \nu_a + \nu_s = \nu.$$

Definition 1.15 (Product measure)

Let $(\Omega_j, \mathcal{A}_j, \mu_j)_{j=1, \dots, n}$ be measure spaces. Define

$$\Omega := \Omega_1 \times \dots \times \Omega_n \text{ and } \mathcal{A} := \bigotimes_{j=1}^n \mathcal{A}_j := \sigma(\{A_1 \times \dots \times A_n \mid A_j \in \mathcal{A}_j\}).$$

We call a measure μ on (Ω, \mathcal{A}) product measure if

$$\mu(A_1 \times \dots \times A_n) = \prod_{j=1}^n \mu_j(A_j), \text{ for all } A_j \in \mathcal{A}_j.$$

Theorem 1.16 (Product measure)

In the former situation if $(\Omega_j, \mathcal{A}_j, \mu_j)_{j=1, \dots, n}$ are σ -finite measure spaces, then the product measure is unique.

Remark 1.11 We denote the product measure by $\mu = \bigotimes_{i=1}^n \mu_i = \mu_1 \otimes \dots \otimes \mu_n$ and call $(\Omega, \mathcal{A}, \mu)$ the product of $(\Omega_j, \mathcal{A}_j, \mu_j)_{j=1, \dots, n}$.

Theorem 1.17 (Tonelli)

In the above situation, let $(\Omega_1, \mathcal{A}_1, \mu_1), (\Omega_2, \mathcal{A}_2, \mu_2)$ be two σ -finite measure spaces and $f : \Omega \rightarrow [0, \infty]$ be a measurable function, then

$$\begin{aligned} \int_{\Omega} f \, d(\mu_1 \otimes \mu_2) &= \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) \, d\mu_1(\omega_1) \right) \, d\mu_2(\omega_2) \\ &= \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) \, d\mu_2(\omega_2) \right) \, d\mu_1(\omega_1). \end{aligned}$$

Theorem 1.18 (Fubini)

In the above situation let $(\Omega_1, \mathcal{A}_1, \mu_1), (\Omega_2, \mathcal{A}_2, \mu_2)$ be two σ -finite measure spaces and $f : \Omega \rightarrow \overline{\mathbb{R}}$ be $\mu_1 \otimes \mu_2$ -integrable, then

1. $f(\omega_1, \cdot)$ is μ_2 -integrable for μ_1 -almost every $\omega_1 \in \Omega_1$,
2. $f(\cdot, \omega_2)$ is μ_1 -integrable for μ_2 -almost every $\omega_2 \in \Omega_2$,
3. the μ_1 -almost everywhere defined functions

$$\omega_1 \mapsto \int_{\Omega_2} f(\omega_1, \omega_2) \, d\mu_2(\omega_2)$$

is μ_1 -integrable,

4. the μ_2 -almost everywhere defined functions

$$\omega_2 \mapsto \int_{\Omega_1} f(\omega_1, \omega_2) \, d\mu_1(\omega_1)$$

is μ_2 -integrable and

- 5.

$$\begin{aligned} \int_{\Omega} f \, d(\mu_1 \otimes \mu_2) &= \int_{\Omega_2} \left(\int_{\Omega_1} f(\omega_1, \omega_2) \, d\mu_1(\omega_1) \right) \, d\mu_2(\omega_2) \\ &= \int_{\Omega_1} \left(\int_{\Omega_2} f(\omega_1, \omega_2) \, d\mu_2(\omega_2) \right) \, d\mu_1(\omega_1). \end{aligned}$$

2 Random Variables

2.1 Independence

Definition 2.1 (Independence)

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We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let I be an arbitrary index set. In the following we define three notions of independence.

1. Independence of sets $A_i \in \mathcal{A}$ with respect to \mathbb{P} :

$$(A_i)_{i \in I} \perp\!\!\!\perp : \iff \forall J \subseteq I, |J| < \infty : \quad \mathbb{P}(\cap_{j \in J} A_j) = \prod_{j \in J} \mathbb{P}(A_j)$$

2. Independence of families of events $\mathcal{F}_i \subseteq \mathcal{A}$ with respect to \mathbb{P} :

$$(\mathcal{F}_i)_{i \in I} \perp\!\!\!\perp : \iff \forall J \subseteq I, |J| < \infty : \quad \mathbb{P}(\cap_{j \in J} A_j) = \prod_{j \in J} \mathbb{P}(A_j), \quad \forall A_j \in \mathcal{F}_j$$

3. Independence of random variables $X_i : (\Omega, \mathcal{A}) \rightarrow (E_i, \mathcal{E}_i)$ with respect to \mathbb{P} :

$$(X_i)_{i \in I} \perp\!\!\!\perp : \iff (\mathcal{I}(X_i) = X_i^{-1}(\mathcal{E}_i))_{i \in I} \perp\!\!\!\perp$$

That is, the initial σ -algebras generated by $(X_i)_{i \in I}$ are independent with respect to \mathbb{P} as families of events.

Lemma 2.1 Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, we have

$$(A_i)_{i \in I} \in \mathcal{A} \perp\!\!\!\perp \iff (\sigma(A_i))_{i \in I} \subseteq \mathcal{A} \perp\!\!\!\perp \iff (\mathbb{1}_{A_i})_{i \in I} \text{ measurable } \perp\!\!\!\perp .$$

Theorem 2.1 Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let $(\mathcal{F}_i)_{i \in I} \subseteq \mathcal{A}$ be \cap -stable, then

$$(\mathcal{F}_i)_{i \in I} \perp\!\!\!\perp \iff (\sigma(\mathcal{F}_i))_{i \in I} \perp\!\!\!\perp .$$

Lemma 2.2 (1st block lemma)

Consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, let

$$\mathcal{F}_{i,k} \subseteq \mathcal{A}, \quad 1 \leq k \leq n(i), \quad 1 \leq i \leq m \in \mathbb{N},$$

be independent \cap -stable families of events. Then the following σ -fields

$$\mathcal{G}_i := \sigma(\mathcal{F}_{i,1} \cup \dots \cup \mathcal{F}_{i,n(i)}), \quad 1 \leq i \leq m,$$

are independent.

Lemma 2.3 (2nd block lemma)

Let

$$X_{i,k} : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), \quad 1 \leq k \leq n(i), \quad 1 \leq i \leq m \in \mathbb{N},$$

be independent random variables and $f_i : E^{n(i)} \rightarrow \mathbb{R}$ measurable functions. Then the random variables

$$z_i = f_i(X_{i1}, \dots, X_{in(i)})$$

are independent.

Theorem 2.2 Let $X_1, \dots, X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E})$ be random variables. Then the following statements are equivalent. ▷ 24.10.2019

1. X_1, \dots, X_n are independent.
2. $\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \prod_{i=1}^n \mathbb{P}(X_i \in A_i), \quad \forall A_1, \dots, A_n \in \mathcal{E}$
3. $\mathbb{P}^{(X_1, \dots, X_n)} = \mathbb{P}^{X_1} \otimes \dots \otimes \mathbb{P}^{X_n}$

(The joint distribution of the random vector is the product of its marginal distributions.)

Corollary 2.2.1 Let $X, Y : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \mathbb{R}^d$ be independent random variables and $h : \mathbb{R}^{2d} \rightarrow \mathbb{R}$ measurable. If $h \geq 0$ or if $h(X, Y)$ is \mathbb{P} -integrable, then

$$\begin{aligned} \mathbb{E}h(X, Y) &= \int \int h(x, y) \mathbb{P}^X(dx) \mathbb{P}^Y(dy) \\ &= \mathbb{E} \int h(x, Y) \mathbb{P}^X(dx) \\ &= \mathbb{E} \int h(X, y) \mathbb{P}^Y(dy). \end{aligned}$$

Corollary 2.2.2 Let $X, Y : \Omega \rightarrow \mathbb{R}^d$ be independent random variables and $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ be measurable functions such that $f, g \geq 0$ or $\mathbb{E}|f(X)|, \mathbb{E}|g(Y)| < \infty$, then

$$\mathbb{E}(f(X) \cdot g(Y)) = (\mathbb{E}f(X)) \cdot (\mathbb{E}g(Y)).$$

Corollary 2.2.3 Let $X_1, \dots, X_n : \Omega \rightarrow \mathbb{R}^d$ be independent random variables and $f_1, \dots, f_n : \mathbb{R}^d \rightarrow \mathbb{R}$ measurable functions such that $f_1, \dots, f_n \geq 0$ or $\mathbb{E}|f_i(X_i)| < \infty$, for all $i = 1, \dots, n$, then

$$\mathbb{E} \left(\prod_{i=1}^n f_i(X_i) \right) = \prod_{i=1}^n \mathbb{E}(f_i(X_i)).$$

Definition 2.2 (Convolution)

Let μ, ν be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then the following measure defines the convolution of μ and ν ▷ 29.10.2019

$$(\mu * \nu)(B) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}_B(x + y) \mu(dx) \nu(dy), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

Theorem 2.3 Let $X, Y : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be independent random variables, then

1. $\mathbb{P}^{X+Y} = \mathbb{P}^X * \mathbb{P}^Y$ and
2. if $\mathbb{P}^X \ll \lambda^d$ with density f and $\mathbb{P}^Y \ll \lambda^d$ with density g , then $\mathbb{P}^{X+Y} \ll \lambda^d$ with density

$$h(z) = \int_{\mathbb{R}^d} f(z - x) \cdot g(x) \lambda^d(dx), \quad z \in \mathbb{R}^d.$$

Remark 2.1 One can regard $\mu * \nu$ as the image measure of $\mu \otimes \nu$ under the map $(x, y) \mapsto x + y$.

Definition 2.3 (Variance, Covariance, Uncorrelated)

Let $X, Y : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables and let X^2, Y^2 be \mathbb{P} -integrable. We define

1. the variance of X as $\mathbb{V}(X) := \mathbb{E}((X - \mathbb{E}X)^2)$ and
2. the covariance of X and Y as $\text{Cov}(X, Y) := \mathbb{E}((X - \mathbb{E}X)(Y - \mathbb{E}Y))$.

If $\text{Cov}(X, Y) = 0$ we call X and Y uncorrelated.

Lemma 2.4 Let $X, Y : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables, let X^2, Y^2 be \mathbb{P} -integrable and $a, b \in \mathbb{R}$, then

1. $\mathbb{V}(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$,
2. $a \mapsto \mathbb{E}(X - a)^2$ is minimized for $a = \mathbb{E}X$,
3. $\mathbb{V}(aX + b) = a^2\mathbb{V}(X)$,
4. $\text{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}X\mathbb{E}Y$ and $\text{Cov}(X, X) = \mathbb{V}(X)$,
5. $\mathbb{V}(X + Y) - \mathbb{V}(X - Y) = 4\text{Cov}(X, Y)$ and
6. $\mathbb{V}(X + Y) = \mathbb{V}(X) + \mathbb{V}(Y) - 2\text{Cov}(X, Y)$.

Theorem 2.4 (Bianaymé's identity)

Let $X_1, \dots, X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be pairwise uncorrelated random variables and let X_1^2, \dots, X_n^2 be \mathbb{P} -integrable, then

$$\mathbb{V}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \mathbb{V}(X_i).$$

2.2 Construction of Independent Random Variables

Problem 1 Given a measurable space (E, \mathcal{E}) and probability measure \mathbb{P} , is there always a random variable X taking values in E and having distribution \mathbb{P} ? ▷ 31.10.2019

Problem 2 Given a measurable space (E, \mathcal{E}) and probability measures $\mathbb{P}_1, \mathbb{P}_2$, are there always random variables $X \sim \mathbb{P}_1, Y \sim \mathbb{P}_2$ taking values in E with $X \perp\!\!\!\perp Y$?

Problem 3 Given (E, \mathcal{E}) and probability measures $(\mathbb{P}_i)_{i \in I}$, where I is an arbitrary index set, are there always random variables $X_i \sim \mathbb{P}_i$ taking values in E with $(X_i)_{i \in I} \perp\!\!\!\perp$?

Solution to problem 1: Let $(E, \mathcal{E}, \mathbb{P})$ be a measure space. Define a random variable

$$X : (E, \mathcal{E}, \mathbb{P}) \rightarrow (E, \mathcal{E}), \omega \mapsto \omega \quad \text{that is} \quad X := \text{id}.$$

It follows that $\mathbb{P}^X = \mathbb{P} \circ \text{id}^{-1} = \mathbb{P}$.

Solution to problem 2: By applying the solution to problem 1, we can find probability spaces $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1), (\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ and random variables

$$X : (\Omega_1, \mathcal{A}_1, \mathbb{P}_1) \rightarrow (E, \mathcal{E}) \quad \text{with} \quad \mathbb{P}_1^X = \mathbb{P}_1,$$

$$Y : (\Omega_2, \mathcal{A}_2, \mathbb{P}_2) \rightarrow (E, \mathcal{E}) \quad \text{with} \quad \mathbb{P}_2^Y = \mathbb{P}_2.$$

Define $\Omega := \Omega_1 \times \Omega_2, \mathcal{A} := \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P} := \mathbb{P}_1 \otimes \mathbb{P}_2$ and new random variables

$$\tilde{X} : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), \omega = (\omega_1, \omega_2) \mapsto X(\omega_1),$$

$$\tilde{Y} : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), \omega = (\omega_1, \omega_2) \mapsto Y(\omega_2).$$

They are well defined, i.e. measurable, as X, Y are measurable. To show independence and by theorem 2.2 it is sufficient to prove that for $B_1, B_2 \in \mathcal{E}$

$$\mathbb{P}(\tilde{X} \in B_1, \tilde{Y} \in B_2) = \mathbb{P}_1 \otimes \mathbb{P}_2(\underbrace{\{\tilde{X} \in B_1\} \cap \{\tilde{Y} \in B_2\}}_{=\{X \in B_1\} \cap \{Y \in B_2\}}) = \mathbb{P}_1(\underbrace{X \in B_1}_{=\{X \in B_1\}}) \mathbb{P}_2(\underbrace{Y \in B_2}_{=\{Y \in B_2\}}).$$

Therefore $\mathbb{P}^{\tilde{X}} = \mathbb{P}_1^X, \mathbb{P}^{\tilde{Y}} = \mathbb{P}_2^Y$ and $\tilde{X} \perp\!\!\!\perp \tilde{Y}$.

Definition 2.4 (Generalized product)

Let $I \neq \emptyset$ be an index set and $(\Omega_i, \mathcal{A}, \mathbb{P}_i)_{i \in I}$ probability spaces. For $\emptyset \neq K \subseteq I$ define

$$\Omega_K := \prod_{k \in K} \Omega_k := \left\{ f : K \rightarrow \bigcup_{k \in K} \Omega_k \mid f(k) \in \Omega_k, \text{ for all } k \in K \right\}$$

as the generalized product of $(\Omega_k)_{k \in K}$. Further we define

- the i -th coordinate of $f \in \Omega_K$ as $f(i), i \in K$,
- the i -th coordinate projection $\pi_i : \Omega_I \rightarrow \Omega_i, f \mapsto f(i), i \in I$,
- the restricted i -th coordinate projection $\pi_i^K : \Omega_K \rightarrow \Omega_i, f \mapsto f(i), K \subseteq I, i \in K$,
- the restriction to Ω_K as $\pi_K^J : \Omega_J \rightarrow \Omega_K, f \mapsto f|_K, K \subseteq J, \pi_K := \pi_K^I$ and
- the set of non-empty finite index subsets of I as $\kappa := \kappa(I) := \{K \subseteq I \mid 0 < |K| < \infty\}$.

Definition 2.5 (Generalized product σ -field)

Let $I \neq \emptyset$ be an index set and let $(\Omega_i, \mathcal{A}_i)_{i \in I}$ be measurable spaces, then

$$\mathcal{A}_I := \bigotimes_{i \in I} \mathcal{A}_i := \mathcal{I}((\pi_i)_{i \in I}),$$

that is, the generalized product σ -field of $(\Omega_i, \mathcal{A}_i)_{i \in I}$ is defined as the smallest σ -field on Ω_I such that all coordinate projections $\pi_i : \Omega_I \rightarrow (\Omega_i, \mathcal{A}_i)$ are measurable, i.e. their initial σ -algebra.

Remark 2.2 The projection maps $\pi_H : \Omega_I \rightarrow \Omega_H, H \in \kappa$ are \mathcal{A}_I - \mathcal{A}_H -measurable. It is possible to verify using the generator of \mathcal{A}_H given by so called cylinder sets of the form

$$\bigcap_{i \in H} (\pi_i^H)^{-1}(A_i), \quad A_i \in \mathcal{A}_i \text{ for all } i \in H.$$

Theorem 2.5 (Kolmogorov existence theorem)

Let $I \neq \emptyset$ be an index set and $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i)_{i \in I}$ probability spaces. Then there exists a unique probability measure

$$\mathbb{P}_I = \bigotimes_{i \in I} \mathbb{P}_i \quad \text{on } (\Omega_I, \mathcal{A}_I)$$

with the property that for all $H \in \kappa$

$$\mathbb{P}_I \circ \pi_H^{-1} = \mathbb{P}_I^{\pi_H} = \bigotimes_{i \in H} \mathbb{P}_i.$$

That is, it exists a unique measure where every finite restriction has an image measure that is equal to the finite product measure.

Definition 2.6 (Generalized product measure, Generalized Product)

The measure \mathbb{P}_I defined above is called the generalized product measure on $(\Omega_I, \mathcal{A}_I)$ and the measure space $(\Omega_I, \mathcal{A}_I, \mathbb{P}_I)$ is called the generalized product of $(\Omega_i, \mathcal{A}_i, \mathbb{P}_i)_{i \in I}$.

Corollary 2.5.1 Let $I \neq \emptyset$ be an index set and let $X_i : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), i \in I$ be random variables, then

$$X_I : (\Omega, \mathcal{A}) \longrightarrow (E_I, \mathcal{E}_I), \omega \longmapsto (i \mapsto X_i(\omega))_{i \in I} \text{ is measurable}$$

and

$$(X_i)_{i \in I} \perp\!\!\!\perp \iff \mathbb{P}^{X_I} = \bigotimes_{i \in I} \mathbb{P}^{X_i},$$

where \mathbb{P}^{X_I} and $\bigotimes_{i \in I} \mathbb{P}^{X_i}$ are both measures on (E_I, \mathcal{E}_I) .

Solution to problem 3: Let $I \neq \emptyset$ be an index set, let (E, \mathcal{E}) be a measure space and $(\mathbb{P}_i)_{i \in I}$ corresponding measures. For each $j \in I$ define a random variable

$$X_j : \left(E_I, \mathcal{E}_I, \bigotimes_{i \in I} \mathbb{P}_i \right) \longrightarrow (E, \mathcal{E}), (\omega : I \rightarrow E) \longmapsto \omega(j),$$

which is equal the projection map π_j . It exists with respect to its unique measure due to the Kolmogorov existence theorem. To these we apply the previous corollary as follows.

As per the previous corollary define

$$X_I : (E_I, \mathcal{E}_I) \longrightarrow (E_I, \mathcal{E}_I), (\omega : I \rightarrow E) \longmapsto (j \mapsto X_j(\omega)) = (\omega : I \rightarrow E),$$

which is equal to the identity map. Therefore, clearly for all $j \in I$

$$\mathbb{P}^{X_j} = \left(\bigotimes_{i \in I} \mathbb{P}_i \right) \circ \pi_j^{-1} \circ = \mathbb{P}_j \quad \text{and} \quad \mathbb{P}^{X_I} = \left(\bigotimes_{i \in I} \mathbb{P}_i \right) \circ \text{id}_{E_I} = \bigotimes_{i \in I} \mathbb{P}_i = \bigotimes_{i \in I} \mathbb{P}^{X_i}.$$

That is, $X_j \sim \mathbb{P}_j$ for all $j \in I$ and $(X_j)_{j \in I} \perp\!\!\!\perp$ as they fulfill the right of the equality in the previous corollary.

Lemma 2.5 (3rd block lemma)

▷ 05.11.2019

Let $I \neq \emptyset$ be an index set, let $X_i : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow (\Omega_i, \mathcal{A}_i), i \in I$ be independent random variables, let $I = \bigcup_{k \in K} I_k$ be a partition of I and

$$f_k : \left(\prod_{i \in I_k} \Omega_i, \bigotimes_{i \in I_k} \mathcal{A}_i \right) \longrightarrow (E_k, \mathcal{E}_k), k \in K$$

are measurable functions. Then $(f_k((X_i)_{i \in I_k}))_{k \in K}$ are independent.

2.3 0-1 laws

Lemma 2.6 (Borel-Cantelli lemma)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}$, then

1. $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty \implies \mathbb{P}(\limsup_{n \rightarrow \infty} A_n) = 0$ and
2. $(A_n)_{n \in \mathbb{N}}$ are pairwise independent and $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty \implies \mathbb{P}(\limsup_{n \rightarrow \infty} A_n) = 1$.

Definition 2.7 (Terminal σ -algebra)

1. Let (Ω, \mathcal{A}) a measurable space and $(\mathcal{A}_n)_{n \in \mathbb{N}} \subseteq \mathcal{A}$ an a sequence of subsets. We define

$$\tau_k((\mathcal{A}_n)_{n \in \mathbb{N}}) := \sigma \left(\bigcup_{m=k}^{\infty} \mathcal{A}_m \right).$$

We call

$$\tau_{\infty}((\mathcal{A}_n)_{n \in \mathbb{N}}) := \bigcap_{k=1}^{\infty} \tau_k((\mathcal{A}_n)_{n \in \mathbb{N}})$$

the terminal σ -algebra associated with $(\mathcal{A}_n)_{n \in \mathbb{N}}$.

2. If we are instead given measurable functions $X_n : (\Omega, \mathcal{A}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}$, we define

$$\tau_k((X_n)_{n \in \mathbb{N}}) := \mathcal{I}((X_m)_{m=k}^{\infty})$$

and call

$$\tau_{\infty}((X_n)_{n \in \mathbb{N}}) := \bigcap_{k=1}^{\infty} \tau_k((X_n)_{n \in \mathbb{N}})$$

the terminal σ -algebra associated with $(X_n)_{n \in \mathbb{N}}$.

Remark 2.3 Setting $\mathcal{A}_n := X_n^{-1}(\mathcal{E}), n \in \mathbb{N}$, one could equivalently define $\tau_k((X_n)_{n \in \mathbb{N}}) := \tau_k((\mathcal{A}_n)_{n \in \mathbb{N}})$.

Theorem 2.6 (Kolmogorov's 0-1-law)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}$ be independent random variables and let $\tau_{\infty}((X_n)_{n \in \mathbb{N}})$ be their associated terminal σ -algebra, then

$$A \in \tau_{\infty}((X_n)_{n \in \mathbb{N}}) \implies \mathbb{P}(A) \in \{0, 1\}.$$

Corollary 2.6.1 (Borel's 0-1-law) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and let $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}$ be independent, then $\mathbb{P}(\limsup_{n \rightarrow \infty} A_n) \in \{0, 1\}$.

Corollary 2.6.2 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}$ be independent random variables, let $\tau_{\infty}((X_n)_{n \in \mathbb{N}})$ be their associated terminal σ -algebra and let $Y : \Omega \rightarrow \mathbb{R}$ be a $\tau_{\infty}((X_n)_{n \in \mathbb{N}})$ - $\mathcal{B}(\mathbb{R})$ -measurable random variable. Then Y is constant \mathbb{P} -almost surely. ▷ 07.11.2019

Definition 2.8 (Finite permutation, Symmetric event)

▷ 12.11.2019

· We call a map $\psi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\{k \in \mathbb{N} \mid \psi(k) \neq k\} < \infty$ a finite permutation and $\Psi := \{\psi : \mathbb{N} \rightarrow \mathbb{N} \mid \psi \text{ is a finite permutation}\}$.

· Let

$$T_{\psi} : (E_{\mathbb{N}}, \mathcal{E}_{\mathbb{N}}) \longrightarrow (E_{\mathbb{N}}, \mathcal{E}_{\mathbb{N}}), (x_n)_{n \in \mathbb{N}} \longmapsto (x_{\psi(n)})_{n \in \mathbb{N}}, \psi \in \Psi,$$

we call an event $A \in \mathcal{E}_{\mathbb{N}}$ symmetric

$$:\iff T_{\psi}(A) = A, \forall \psi \in \Psi$$

and $\mathcal{E}_{\text{sym}} := \{A \in \mathcal{E}_{\mathbb{N}} \mid A \text{ symmetric}\}$.

Lemma 2.7 Let $(E_{\mathbb{N}}, \mathcal{E}_{\mathbb{N}})$ be a generalized product space, then the set of symmetric events $\mathcal{E}_{\text{sym}} \subseteq \mathcal{E}_{\mathbb{N}}$ is a σ -algebra over $E_{\mathbb{N}}$. ▷ 14.11.2019

Lemma 2.8 Let $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ be σ -fields over Ω , let $A \in \mathcal{A} := \sigma(\bigcup_{i=1}^{\infty} \mathcal{A}_i)$ and let \mathbb{P} be a probability measure on (Ω, \mathcal{A}) . Then, there exists $A_i \in \mathcal{A}_i, \forall i \in \mathbb{N}$, such that

$$\mathbb{P}((A \setminus A_i) \cup (A_i \setminus A)) \xrightarrow{i \rightarrow \infty} 0.$$

Theorem 2.7 (Hewitt-Savage 0-1-law)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}$ be a family independent and identically distributed random variables and define

$$X_{\mathbb{N}} : (\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow (E_{\mathbb{N}}, \mathcal{E}_{\mathbb{N}}), \omega \longmapsto (X_n(\omega))_{n \in \mathbb{N}},$$

with values in the associated generalized product. Then,

$$\mathbb{P}^{X_{\mathbb{N}}}(A) \in \{0, 1\}, \forall A \in \mathcal{E}_{\text{sym}}, \text{ where } \mathcal{E}_{\text{sym}} \subseteq \mathcal{E}_{\mathbb{N}}.$$

That is, symmetric events fulfill a 0-1-law with respect to the image measure of the random variables $(X_n)_{n \in \mathbb{N}}$.

2.4 Strong Law of Large Numbers

Theorem 2.8 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})), n \in \mathbb{N}$ be independent and identically distributed random variables with $\mathbb{E}|X_1| = \infty$. Then for $S_n := X_1 + \dots + X_n, n \in \mathbb{N}$, we have

1. $\mathbb{P}(\limsup_{n \rightarrow \infty} \{|X_n| \geq n\}) = 1$ and
2. $\mathbb{P}(\{\lim_{n \rightarrow \infty} \frac{S_n}{n} \text{ exists and is finite}\}) = 0$.

Definition 2.9 (Almost sure convergence)

▷ 19.11.2019

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})), n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ be random variables, we write

$$X_n \xrightarrow{\text{a.s.}} X \iff \lim_{n \rightarrow \infty} X_n = X \text{ } \mathbb{P}\text{-almost surely} \iff \mathbb{P}\left(\left\{\lim_{n \rightarrow \infty} X_n = X\right\}\right) = 1$$

and say $(X_n)_{n \in \mathbb{N}}$ converge \mathbb{P} -almost surely to X .

Lemma 2.9 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})), n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$ be random variables, then

$$X_n \xrightarrow{\text{a.s.}} X \iff \mathbb{P}\left(\limsup_{n \rightarrow \infty} \{|X_n - X| > \varepsilon\}\right) = 0, \text{ for all } \varepsilon > 0.$$

Lemma 2.10 (Cesàro's lemma)

Let $(a_n)_{n \in \mathbb{N}} \in (0, \infty)$ such that $a_n \uparrow \infty$ and let $(v_k)_{k \in \mathbb{N}} \in \mathbb{R}$ be convergent with $\lim_{k \rightarrow \infty} v_k = v_{\infty}$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{a_n} \sum_{k=1}^n (a_k - a_{k-1})v_k = v_{\infty}, \quad a_0 := 0.$$

Lemma 2.11 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}})), n \in \mathbb{N}$ be independent and identically distributed random variables with $\mathbb{E}|X_1| < \infty$. Define

$$Y_n := X_n \cdot \mathbb{1}_{\{|X_n| \leq n\}}, \quad T_n := Y_1 + \dots + Y_n.$$

It follows, that

1. $\lim_{n \rightarrow \infty} \mathbb{E}Y_n = \mathbb{E}X_1$,

2. $\mathbb{P}(\limsup_{n \rightarrow \infty} \{X_n \neq Y_n\}) = 0$,
3. $\sum_{n=1}^{\infty} \frac{\mathbb{V}Y_n}{n^2} \leq 2\mathbb{E}|X_1|$ and
4. $\sum_{n=1}^{\infty} \frac{\mathbb{V}T_{\lfloor \alpha^n \rfloor}}{\lfloor \alpha^n \rfloor^2} \leq \frac{2\alpha}{\alpha-1} \mathbb{E}|X_1|$ for all $\alpha > 1$.

Theorem 2.9 (Strong law of large numbers)

▷ 21.11.2019

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, $n \in \mathbb{N}$ be independent and identically distributed random variables with $\mathbb{E}|X_1| < \infty$. Then

$$\frac{X_1 + \dots + X_n}{n} \xrightarrow{\text{a.s.}} \mathbb{E}X_1.$$

Definition 2.10 (Empirical distribution function)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, $n \in \mathbb{N}$ be independent and identically distributed random variables. Define the empirical distribution function associated with $(X_n)_{n \in \mathbb{N}}$ as

$$F_n(t) := \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{(-\infty, t]}(X_i), \quad n \in \mathbb{N}.$$

Corollary 2.9.1 (Glivenko-Cantelli theorem)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$, $n \in \mathbb{N}$ be independent and identically distributed random variables with distribution function F and let F_n be the associated empirical distribution function. Then

$$\lim_{n \rightarrow \infty} \sup_{t \in \mathbb{R}} |F_n(t) - F(t)| = 0 \quad \mathbb{P}\text{-almost surely.}$$

3 Characteristic Functions

Definition 3.1 (Characteristic function)

▷ 26.11.2019

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be a random variable. We define

$$\varphi_X : \mathbb{R}^d \rightarrow \mathbb{C}, t \mapsto \mathbb{E} \left(e^{i\langle t, X \rangle} \right)$$

as the characteristic function of the random variable X .

Theorem 3.1 Let $G \sim \mathcal{N}(0, 1)$, then $\varphi_G(t) = e^{-\frac{t^2}{2}}, t \in \mathbb{R}$.

Theorem 3.2 Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable and $a, b \in \mathbb{R}$. Then

$$\varphi_{aX+b}(t) = e^{itb} \varphi_X(at), t \in \mathbb{R}.$$

Corollary 3.2.1 $X \sim \mathcal{N}(\mu, \sigma^2) \implies \varphi_X(t) = e^{it\mu - \frac{\sigma^2 t^2}{2}}, t \in \mathbb{R}$.

Theorem 3.3 (Properties of characteristic functions)

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be a random variable with characteristic function φ_X . Then

1. φ_X is uniformly continuous,
2. $\|\varphi_X\|_\infty \leq 1$,
3. $\varphi_{-X}(t) = \varphi_X(-t) = \overline{\varphi_X(t)}$ for all $t \in \mathbb{R}^d$,
4. $t \mapsto \operatorname{Re} \varphi_X(t) = \mathbb{E} \cos\langle t, X \rangle$ is the characteristic function of $\varepsilon \cdot X$, where $\varepsilon : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow \{-1, 1\}$, $\mathbb{P}(\varepsilon = \pm 1) = \frac{1}{2}$ and $\varepsilon \perp\!\!\!\perp X$,
5. $|\varphi_X(t)|^2$ is the characteristic function of $X - \tilde{X}$, where X, \tilde{X} are independent and identically distributed,
6. let

$$T_X = \Sigma X + m, \Sigma \in \mathbb{R}^{n \times d} \text{ and } m \in \mathbb{R}^n,$$

it follows that

$$\varphi_{T_X}(t) = e^{i\langle t, m \rangle} \cdot \varphi_X(\Sigma^T t), t \in \mathbb{R}^d,$$

7. let $d = 1$ and $\mathbb{E}|X|^n < \infty$ for some $n \in \mathbb{N}$, it follows that φ_X is n -times continuously differentiable and

$$\varphi_X^{(k)}(0) = i^k \cdot \mathbb{E} X^k, \text{ for all } k \in \{0, \dots, n\} \text{ and}$$

8. let $d = 1$, $t_1, \dots, t_n \in \mathbb{R}$ and $A_\varphi := (a_{ij})_{i,j=1, \dots, n}$, where $a_{ij} := \varphi_X(t_i - t_j)$, it follows that A_φ is Hermitian and positive semidefinite.

Theorem 3.4 (Bodiner's theorem)

▷ 28.11.2019

Let $\varphi : \mathbb{R} \rightarrow \mathbb{C}$, then the following statements are equivalent.

1. φ is the characteristic function of a \mathbb{R} -valued random variable.
2. φ is continuous, $\varphi(0) = 1$ and $A_\varphi := (a_{ij})_{i,j=1, \dots, n}$ is positive semidefinite, where $a_{ij} := \varphi_X(t_i - t_j)$, $t_1, \dots, t_n \in \mathbb{R}$.

Theorem 3.5 (Lévy's theorem)

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable and $[a, b] \subset \mathbb{R}$, then

$$\frac{1}{2}\mathbb{P}(X = a) + \frac{1}{2}\mathbb{P}(X = b) + \mathbb{P}(a < X < b) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-ita} - e^{-itb}}{it} \varphi(t) dt.$$

Corollary 3.5.1 Let $X : (\Omega_1, \mathcal{A}_1, \mathbb{P}_1) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $Y : (\Omega_2, \mathcal{A}_2, \mathbb{P}_2) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables with $\varphi_X = \varphi_Y$, then $X \stackrel{d}{=} Y$.

Corollary 3.5.2 (Kac's theorem)

Let $X, Y : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables, then

$$X \perp\!\!\!\perp Y \iff \varphi_{(X,Y)}(s, t) = \varphi_X(s)\varphi_Y(t), \quad s, t \in \mathbb{R}.$$

Theorem 3.6 (Lévy's inequality)

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be a random variable and $r > 0$. Then

$$\mathbb{P}\left(\max_{1 \leq k \leq d} |X_k| \geq r\right) \leq 7 \left(\frac{r}{2}\right)^d \int_{-\frac{1}{r}}^{\frac{1}{r}} \cdots \int_{-\frac{1}{r}}^{\frac{1}{r}} (1 - \operatorname{Re} \varphi_X(t)) dt_1 \dots dt_d.$$

4 Convergence of Random Variables

▷ 03.12.2019

Definition 4.1 (Modes of convergence)

Let $X_n : (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. We define the following modes of convergence.

1. Convergence *in distribution*:

$$X_n \xrightarrow{d} X \iff \forall f \in \mathcal{C}_b(\mathbb{R}) : \lim_{n \rightarrow \infty} \mathbb{E}f(X_n) = \mathbb{E}f(X)$$

Let now $\Omega = \Omega_1 = \dots = \Omega_n$, $\mathcal{A} = \mathcal{A}_1 = \dots = \mathcal{A}_n$ and $\mathbb{P} = \mathbb{P}_1 = \dots = \mathbb{P}_n$.

2. Convergence *almost surely*:

$$X_n \xrightarrow{\text{a.s.}} X \iff \mathbb{P}(\{\omega \in \Omega \mid \lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)\}) = 1$$

3. Convergence *in L^p /in p^{th} mean*, $1 < p \in \mathbb{R}$:

$$X_n \xrightarrow{L^p} X \iff X, X_n \in L^p \text{ and } \lim_{n \rightarrow \infty} \|X_n - X\|_p = 0$$

4. Convergence *in probability*:

$$X_n \xrightarrow{\mathbb{P}} X \iff \forall \epsilon > 0 : \lim_{n \rightarrow \infty} \mathbb{P}(|X_n - X| > \epsilon) = 0$$

Remark 4.1 For a real valued random variable Z , let $\|Z\|_p := (\mathbb{E}|Z|^p)^{1/p}$, $p \in \mathbb{N}$. Further, let $\mathcal{C}_b(\mathbb{R})$ denote the set of continuous and bounded functions on \mathbb{R} to \mathbb{R} .

Lemma 4.1 (Uniqueness of limits) Let $X_n : (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $Y : (E, \mathcal{E}, \mathbb{W}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. It follows that

1. $X_n \xrightarrow{d} X$ and $X_n \xrightarrow{d} Y \implies X \stackrel{d}{=} Y$.

Let now $\Omega = E = \Omega_1 = \dots = \Omega_n$, $\mathcal{A} = \mathcal{E} = \mathcal{A}_1 = \dots = \mathcal{A}_n$ and $\mathbb{P} = \mathbb{W} = \mathbb{P}_1 = \dots = \mathbb{P}_n$. Then

2. $X_n \xrightarrow{\text{a.s.}} X$ and $X_n \xrightarrow{\text{a.s.}} Y \implies X = Y$ \mathbb{P} -almost surely,

3. $X_n \xrightarrow{\mathbb{P}} X$ and $X_n \xrightarrow{\mathbb{P}} Y \implies X = Y$ \mathbb{P} -almost surely,

4. $X_n \xrightarrow{L^p} X$ and $X_n \xrightarrow{L^p} Y \implies X = Y$ \mathbb{P} -almost surely.

Theorem 4.1 (Relating the modes of convergence)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. It follows that

1. $X_n \xrightarrow{\mathbb{P}} X \implies X_n \xrightarrow{d} X$ and $f(X_n) \xrightarrow{L^1} f(X)$, for all $f \in \mathcal{C}_b(\mathbb{R})$,

2. $X_n \xrightarrow{L^p} X$ for some $p > 1 \implies X_n \xrightarrow{L^1} X$,

3. $X_n \xrightarrow{L^1} X \implies X_n \xrightarrow{\mathbb{P}} X$ and

4. $X_n \xrightarrow{\text{a.s.}} X \implies X_n \xrightarrow{\mathbb{P}} X$.

Lemma 4.2 (Fast convergence)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. Assume that $X_n \xrightarrow{\mathbb{P}} X$ and there exist $(\varepsilon_n)_{n \in \mathbb{N}} \in \mathbb{R}$, $\varepsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - X| > \varepsilon_n) < \infty.$$

Then $X_n \xrightarrow{\text{a.s.}} X$.

Corollary 4.1.1 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables. Assume $X_n \xrightarrow{\mathbb{P}} X$ then there is a monotone sequence $(n_k)_{k \in \mathbb{N}} \in \mathbb{N}$ such that $X_{n_k} \xrightarrow{\text{a.s.}} X$.

Lemma 4.3 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables, where $X = b \in \mathbb{R}$ \mathbb{P} -almost surely. Then

$$X_n \xrightarrow{\mathbb{P}} X \iff X_n \xrightarrow{d} X.$$

Remark 4.2 In general, the implications in the below diagram can not be reversed, which can be proved by finding counterexamples.

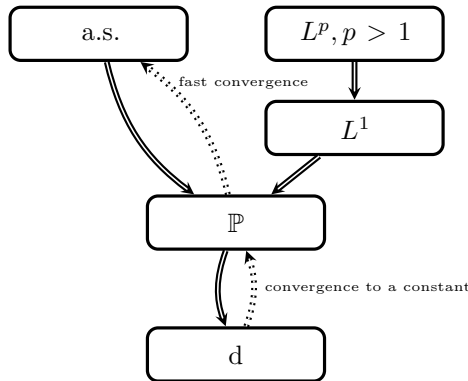


Figure 1: Hierarchy of modes of convergences

Theorem 4.2 Let $X_n : (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables, then ▷ 10.12.2019

$$X_n \xrightarrow{d} X \iff F_{X_n}(t) \xrightarrow{n \rightarrow \infty} F_X(t)$$

for all continuity points t of F_X , where F_{X_n}, F_X are the associated distribution functions.

Lemma 4.4 (Tightness)

Let $X_n : (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $n \in \mathbb{N}$ be random variables. Suppose that the sequence of characteristic functions of $(X_n)_{n \in \mathbb{N}}$ evaluated at t , $(\varphi_{X_n}(t))_{n \in \mathbb{N}}$, has a limit $\varphi(t)$ for all $t \in \mathbb{R}^d$. Then

$$\forall \varepsilon > 0 : \exists r(\varepsilon) : \forall R > r(\varepsilon) : \sup_{n \in \mathbb{N}} \mathbb{P}(\|X_n\| > R) \leq \varepsilon.$$

Theorem 4.3 Let $X_n : (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be random variables. Then

$$X_n \xrightarrow{d} X \iff \varphi_{X_n}(t) \xrightarrow{n \rightarrow \infty} \varphi_X(t) \text{ for all } t \in \mathbb{R}^d.$$

Corollary 4.3.1 (Cramér-Wold theorem)

▷ 12.12.2019

Let $X_n : (\Omega_n, \mathcal{A}_n, \mathbb{P}_n) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $n \in \mathbb{N}$ and $X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ be random variables. Then

$$X_n \xrightarrow{d} X \iff \langle X_n, t \rangle \xrightarrow{d} \langle X, t \rangle \text{ for all } t \in \mathbb{R}^d.$$

Corollary 4.3.2 (Slutsky's lemma)

Let $X_n, Y_n, X : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be random variables with $X_n \xrightarrow{d} X$, $X_n - Y_n \xrightarrow{\mathbb{P}} 0$. Then $Y_n \xrightarrow{d} X$.

5 Central Limit Theorems

5.1 A Look at Several Central Limit Theorems

Definition 5.1 (Central limit theorem)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be random variables and $\mathcal{S}_n := X_1 + \dots + X_n$. Then $(X_n)_{n \in \mathbb{N}}$ satisfies a central limit theorem

$$:\iff \exists (a_n)_{n \in \mathbb{N}}, (s_n)_{n \in \mathbb{N}} \in \mathbb{R} : \frac{\mathcal{S}_n - a_n}{s_n} \xrightarrow{d} G \sim \mathcal{N}(0, 1). \quad (\text{CLT})$$

Theorem 5.1 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be independent and identically distributed, $\mathbb{E}X_1 = 0$, $\mathbb{V}X_1 = \sigma^2 \in (0, \infty)$ and $\mathcal{S}_n := X_1 + \dots + X_n$. Then $(X_n)_{n \in \mathbb{N}}$ satisfies a central limit theorem with $a_n = 0$, $s_n = \sigma\sqrt{n}$, $n \in \mathbb{N}$, that is

$$\frac{\mathcal{S}_n}{\sigma\sqrt{n}} \xrightarrow{d} G \sim \mathcal{N}(0, 1).$$

Theorem 5.2 (Lindeberg (1922) & Lévy (1925, 1937))

▷ 17.12.2019

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be independent random variables. Assume $\mathbb{E}X_n = 0$, $\mathbb{V}X_n = \sigma_n^2 \in (0, \infty)$ and denote the distribution of X_n by μ_n . Define $s_n^2 := \sigma_1^2 + \dots + \sigma_n^2$ and suppose that the Lindeberg condition

$$\forall \varepsilon > 0 : \lim_{n \rightarrow \infty} \frac{1}{s_n^2} \sum_{k=1}^n \int_{\{|x| > \varepsilon s_k\}} x^2 \mu_k(dx) = 0 \quad (\text{L})$$

holds. Let $\mathcal{S}_n := X_1 + \dots + X_n$, then

$$\frac{\mathcal{S}_n}{s_n} \xrightarrow{d} G \sim \mathcal{N}(0, 1).$$

Remark 5.1 In the above setting one can consider the classical Lindeberg condition

$$\forall \varepsilon > 0 : \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{\{|x| > \varepsilon s_n\}} x^2 \mu_k(dx) = 0, \quad (\text{L}')$$

which is equivalent to the Lindeberg condition (L).

Lemma 5.1 (Feller (1935))

In the above setting the Lindeberg condition implies the Feller condition

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq n} \frac{\sigma_k}{s_n} = 0. \quad (\text{F})$$

Corollary 5.2.1 (Lyapunov (1901))

▷ 19.12.2019

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be independent random variables with $\mathbb{E}X_n = 0$, $\mathbb{V}X_n = \sigma_n^2$. Define $s_n^2 := \sigma_1^2 + \dots + \sigma_n^2$ and suppose that the Lyapunov condition

$$\exists \delta > 0 : \lim_{n \rightarrow \infty} \frac{1}{s_n^{2+\delta}} \sum_{k=1}^n \mathbb{E}|X_k|^{2+\delta} = 0 \quad (\text{LY})$$

holds, then (CLT) holds.

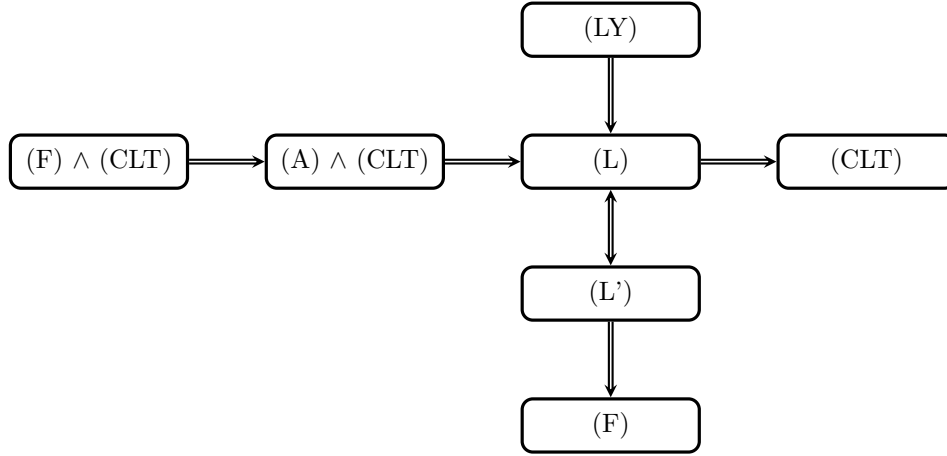


Figure 2: Relations of limit conditions

Remark 5.2 (LY) is usually easier to check than (L), but it is stronger than (L). In practice (LY) is checked with $\delta = 1$.

Theorem 5.3 (Feller (1935))

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be independent random variables with $\mathbb{E}X_n = 0, \forall X_n < \infty, \forall n \in \mathbb{N}$. Then, $(F) \wedge (CLT) \Leftrightarrow (L)$.

Remark 5.3 The results remains valid for triangular arrays of random variables.

$$\begin{array}{ccccccc}
 X_{1,1} & X_{1,2} & \dots & X_{1,k(1)} & & & \\
 X_{2,1} & X_{2,2} & \dots & \dots & X_{2,k(2)} & & \\
 \vdots & \vdots & & & & \ddots & \\
 X_{n,1} & X_{n,2} & \dots & \dots & \dots & \dots & X_{n,k(n)} \\
 \vdots & \vdots & & & & & \ddots
 \end{array}$$

We assume independence within each line, but not of the lines. Roughly, there will be the following correspondence.

classical	triangular arrays
$X_j \sim \mu_j$	$X_{n,j} \sim \mu_{n,j}$
X_j independent	$(X_{n,j})_{j=1, \dots, k(n)}$ independent
$\sigma_j^2 = \mathbb{V}X_j$	$\sigma_{n,j}^2 = \mathbb{V}X_{n,j}$
$s_n^2 = \sum_{j=1}^n \sigma_j^2$	$s_n^2 = \sum_{j=1}^{k(n)} \sigma_{n,j}^2$
$\mathcal{S}_n = \sum_{j=1}^n X_j$	$\mathcal{S}_n = \sum_{j=1}^{k(n)} X_{n,j}$
(L) \vee (L')	$\forall \varepsilon > 0 : \frac{1}{s_n^2} \sum_{j=1}^{k(n)} \int_{ X_j > \varepsilon(s_j \vee s_n)} x^2 \mu_{n,j}(dx) \rightarrow 0$
(F)	$\max_{1 \leq j \leq k(n)} \frac{\sigma_{n,j}}{s_n} \rightarrow 0$
(A)	$\forall \varepsilon > 0 : \max_{1 \leq j \leq k(n)} \mathbb{P}(X_{n,j} > \varepsilon s_n) \rightarrow 0$
(CLT)	$\frac{\mathcal{S}_n - \mathbb{E}\mathcal{S}_n}{s_n} \xrightarrow[n \rightarrow \infty]{d} G \sim \mathcal{N}(0, 1)$

Theorem 5.4 (Goncharov (1944))

Let \mathcal{S}_n be the number of cycles of a uniform random permutation on n elements. Then

$$\frac{\mathcal{S}_n - \log n}{\sqrt{\log n}} \xrightarrow{d} G \sim \mathcal{N}(0, 1).$$

5.2 Poisson Limit Theorem**Theorem 5.5 (Poisson limit theorem)**

Let $X_{1,n}, \dots, X_{n,n} : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B})$, $n \in \mathbb{N}$ be independent random variables such that

$$\mathbb{P}(X_{k,n} = 1) = p_{k,n} \quad \text{and} \quad \mathbb{P}(X_{k,n} = 0) = 1 - p_{k,n} \quad \text{for all } k, n \in \mathbb{N}, 1 \leq k \leq n.$$

Suppose that $\max_{1 \leq k \leq n} p_{k,n} \xrightarrow{n \rightarrow \infty} 0$ and $\sum_{k=1}^n p_{k,n} \xrightarrow{n \rightarrow \infty} \lambda$. Then

$$\mathcal{S}_n = X_{1,n} + \dots + X_{n,n} \xrightarrow{d} Z \sim \text{Po}(\lambda).$$

5.3 Weak Law of Large Numbers**Theorem 5.6 (Weak law of large numbers)**

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be independent and identically distributed random variables with $\mathbb{E}X_1 < \infty$, in the improper Riemann sense, and $\mathcal{S}_n := X_1 + \dots + X_n$. Then $\frac{\mathcal{S}_n}{n} \xrightarrow{\mathbb{P}} \mathbb{E}X_1$.

Remark 5.4 If additionally $\forall X_n < \infty$ for all $n \in \mathbb{N}$, the random variables $(X_n)_{n \in \mathbb{N}}$ need not be independent nor identically distributed, but only pairwise uncorrelated for

$$\frac{1}{n} \sum_{k=1}^n (X_k - \mathbb{E}X_k) \xrightarrow{\mathbb{P}} 0$$

to hold.

6 Conditional Expectations

6.1 Construction and Elementary Properties

Definition 6.1 (Conditional expected value)

▷ 09.01.2020

Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, $F \in \mathcal{A}$ with $\mathbb{P}(F) > 0$. We define the conditional expected value as

$$\mathbb{E}(X | F) := \int X \, d\mathbb{P}_F.$$

Lemma 6.1 Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, $F \in \mathcal{A}$ with $\mathbb{P}(F) > 0$. Then

$$\mathbb{E}(X | F) = \frac{\mathbb{E}(X \cdot \mathbb{1}_F)}{\mathbb{P}(F)}.$$

Definition 6.2 (Conditional expectation)

Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F} \subseteq \mathcal{A}$ be a σ -algebra over Ω . The conditional expectation of X with respect to \mathcal{F} is a \mathcal{F} -measurable random variable $X^{\mathcal{F}} : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ for which either of

$$\forall F \in \mathcal{F} : \int_F X \, d\mathbb{P} = \int_F X^{\mathcal{F}} \, d\mathbb{P} \iff \mathbb{E}(X \mathbb{1}_F) = \mathbb{E}(X^{\mathcal{F}} \mathbb{1}_F) \quad (\text{CE})$$

holds.

Remark 6.1 In the above situation, we will sometimes denote a random variable $X^{\mathcal{F}}$ fulfilling either condition (CE) as

$$\mathbb{E}(X | \mathcal{F}) := X^{\mathcal{F}}.$$

Further, if $X = \mathbb{1}_A$ for $A \in \mathcal{A}$, then

$$\mathbb{P}(A | \mathcal{F}) := \mathbb{E}(\mathbb{1}_A | \mathcal{F}) = X^{\mathcal{F}}.$$

Theorem 6.1 (Existence & uniqueness of conditional expectations)

Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F} \subseteq \mathcal{A}$ be a σ -algebra over Ω .

a) If $X^{\mathcal{F}}, Y^{\mathcal{F}} : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ satisfy

$$\mathbb{E}[X \mathbb{1}_F] = \mathbb{E}[X^{\mathcal{F}} \mathbb{1}_F] = \mathbb{E}[Y^{\mathcal{F}} \mathbb{1}_F], \quad \forall F \in \mathcal{F},$$

then $X^{\mathcal{F}} = Y^{\mathcal{F}}$ \mathbb{P} -almost surely.

b) A random variable $X^{\mathcal{F}} : (\Omega, \mathcal{F}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ fulfilling (CE) exists.

Remark 6.2 When working with conditional expectations, we usually apply the following method. ▷ 14.01.2020

1. Check the defining equation (CE) to find a candidate for the conditional expectation.
2. Conclude by uniqueness, that this candidate is indeed the conditional expectation.

Theorem 6.2 Let $X, Y \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, $a, b, c \in \mathbb{R}$ and let $\mathcal{F} \subseteq \mathcal{A}$ be a σ -algebra over Ω . Then

1. $X \geq 0 \implies \mathbb{E}(X | \mathcal{F}) \geq 0,$
2. $X \equiv c \implies \mathbb{E}(X | \mathcal{F}) \equiv c,$
3. $\mathcal{F} = \{\emptyset, \Omega\} \implies \mathbb{E}(X | \mathcal{F}) = \mathbb{E}X,$
4. $\mathbb{E}(\mathbb{E}(X | \mathcal{F})) = \mathbb{E}X,$
5. $\mathbb{E}(a \cdot X + bY | \mathcal{F}) = a \cdot \mathbb{E}(X | \mathcal{F}) + b \cdot \mathbb{E}(Y | \mathcal{F})$ and
6. $X \geq Y \implies \mathbb{E}(X | \mathcal{F}) \geq \mathbb{E}(Y | \mathcal{F}).$

Theorem 6.3 (Convergence theorems)

Let $X, X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be random variables and $\mathcal{F} \subseteq \mathcal{A}$ a σ -algebra over Ω .

- *Conditional Fatou's lemma: If $X_n \geq 0$, $X_n \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ for all $n \in \mathbb{N}$ and $\liminf_{n \rightarrow \infty} \mathbb{E}X_n < \infty$, then \mathbb{P} -almost surely*

$$\mathbb{E} \left(\liminf_{n \rightarrow \infty} X_n \mid \mathcal{F} \right) \leq \liminf_{n \rightarrow \infty} \mathbb{E}(X_n \mid \mathcal{F}) .$$

- *Conditional dominated convergence: If $X_n \xrightarrow{\text{a.s.}} X$ and there is $Y \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ such that for all $n \in \mathbb{N}$, we have $|X_n| \leq Y$, then \mathbb{P} -almost surely*

$$\lim_{n \rightarrow \infty} \mathbb{E}(X_n \mid \mathcal{F}) = \mathbb{E} \left(\lim_{n \rightarrow \infty} X_n \mid \mathcal{F} \right) = \mathbb{E}(X \mid \mathcal{F}) .$$

- *Conditional monotone convergence: If $X_n \geq 0$, $X_n \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ for all $n \in \mathbb{N}$, $X_n \uparrow X$ and $\sup_{n \in \mathbb{N}} \mathbb{E}X_n < \infty$, then \mathbb{P} -almost surely*

$$\mathbb{E}(X_n \mid \mathcal{F}) \uparrow \mathbb{E}(X \mid \mathcal{F}) .$$

- *Conditional Jensen inequality: If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ convex and measurable and $\mathbb{E}|\varphi(X)| < \infty$, then almost surely*

$$\varphi(\mathbb{E}(X \mid \mathcal{F})) \leq \mathbb{E}(\varphi(X) \mid \mathcal{F}) .$$

Theorem 6.4 Let $X, Y, Z : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables, $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{F} \subseteq \mathcal{A}$ a σ -algebra over Ω , then

1. pull-out: $Z \in L^\infty(\mathcal{F}, \mathbb{P}) \Rightarrow \mathbb{E}(X \cdot Z \mid \mathcal{F}) = Z \cdot \mathbb{E}(X \mid \mathcal{F})$,
2. pull-out: $Z \mathcal{F}$ -mb, $\mathbb{E}|X \cdot Z| < \infty \Rightarrow \mathbb{E}(XZ \mid \mathcal{F}) = Z \cdot \mathbb{E}(X \mid \mathcal{F})$,
3. tower: $\mathcal{G} \subseteq \mathcal{F} \subseteq \mathcal{A}$ σ -fields $\Rightarrow \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}) \mid \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G})$,
4. projection: $X \in L^2(\Omega, \mathcal{A}, \mathbb{P})$, $Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \Rightarrow$
 $\mathbb{E}(X \mid \mathcal{F}) \in L^2(\Omega, \mathcal{A}, \mathbb{P})$, $\mathbb{E} \left[(X - \mathbb{E}(X \mid \mathcal{F}))^2 \right] \leq \mathbb{E}((X - Y)^2)$ and
5. monotonicity: $X \leq Y \Rightarrow \mathbb{E}(X \mid \mathcal{F}) \leq \mathbb{E}(Y \mid \mathcal{F})$.

Remark 6.3 In the above situation, there is a geometric meaning to (4.). $L^2(\Omega, \mathcal{A}, \mathbb{P})$ is a Hilbert space, where $\langle u, v \rangle := \mathbb{E}(uv)$ with $L^2(\Omega, \mathcal{F}, \mathbb{P})$ being a linear subspace. Thus, as per (4.), $\mathbb{E}(X \mid \mathcal{F})$ is the orthogonal projection of X onto $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

6.2 Conditional Expectation and Independence

Theorem 6.5 $X, Y \in L^1(\Omega, \mathcal{A}, \mathbb{P})$, \mathcal{F}, \mathcal{G} σ -algebras over Ω and $g : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^d)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ a bounded and measurable function. Then

▷ 16.01.2020

1. $\mathcal{I}(X) \perp \mathcal{F} \Rightarrow \mathbb{E}(X \mid \mathcal{F}) = \mathbb{E}X$,
2. $\mathcal{I}(X), \mathcal{G} \perp \mathcal{F} \Rightarrow \mathbb{E}(X \mid \mathcal{F}, \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G})$ and
3. $X \perp Y$ and $Y \mathcal{F}$ -measurable $\Rightarrow \mathbb{E}(g(X, Y) \mid \mathcal{F}) = \mathbb{E}g(X, t)|_{t=Y}$.

6.3 Conditioning on Random Variables

Remark 6.4 Let $X : (\Omega, \mathcal{A}) \rightarrow (E, \mathcal{E})$ be measurable and $\mathcal{F}, \mathcal{G} \subseteq \mathcal{A}$ σ -algebras over Ω . We will use abbreviations such as $X \perp\!\!\!\perp \mathcal{F}$ instead of $\mathcal{I}(X) \perp\!\!\!\perp \mathcal{F}$, and similarly $X, \mathcal{G} \perp\!\!\!\perp \mathcal{F}$ instead of $\sigma(\mathcal{I}(X) \cup \mathcal{G}) \perp\!\!\!\perp \mathcal{F}$.

Remark 6.5 Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $Y : (\Omega, \mathcal{A}) \rightarrow (E, \mathcal{E})$ measurable. We use the abbreviation $\mathbb{E}(X | Y) := \mathbb{E}(X | \mathcal{I}(Y))$.

Lemma 6.2 (Factorization Lemma)

$$\left. \begin{array}{l} Y : (\Omega, \mathcal{A}) \xrightarrow{\text{mb.}} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \\ Z : (\Omega, \mathcal{I}(Y)) \xrightarrow{\text{mb.}} (\mathbb{R}, \mathcal{B}(\mathbb{R})) \end{array} \right\} \implies \begin{array}{l} \exists g : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ measurable} \\ \text{such that } Z = g(Y) \end{array}$$

Definition 6.3 Let $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ and $Y : (\Omega, \mathcal{A}) \rightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Then we denote by

$$\mathbb{E}[X | Y = y] := g(y) \in \mathbb{R} \longleftarrow y \in \mathbb{R}^d$$

the measurable function $g : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that satisfies $\mathbb{E}(X | Y) = g(Y)$ by the previous lemma.

Remark 6.6 In the discrete setting, that is $\text{im } Y$ is countable, we are able to evaluate the expression as

$$\mathbb{E}[X | Y = y] = \begin{cases} \frac{\mathbb{E}(X \mathbb{1}_y(Y))}{\mathbb{P}(Y=y)} & \text{if } \mathbb{P}(Y = y) > 0 \\ 0 & \text{otherwise} \end{cases}.$$

What about the continuous setting (conditioning on events with probability 0)?

Definition 6.4 (Conditional density)

Let $(X, Y) : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ be a random variable with an image measure that has a density $f_{(X,Y)}$ with respect to the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. We define the conditional density of X given Y as

$$f_{X|Y}(x, y) := \begin{cases} \frac{f_{(X,Y)}(x, y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{otherwise} \end{cases}, \text{ where } x, y \in \mathbb{R}.$$

Remark 6.7 Recall that in the setting of the previous definition, the following holds \mathbb{P} -almost surely

$$f_Y(y) = \int_{\mathbb{R}} f_{(X,Y)}(x, y) \, dx, \text{ where } y \in \mathbb{R}.$$

Theorem 6.6 Let $(X, Y) : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ be a random variable with an image measure that has a density $f_{(X,Y)}$ with respect to the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ and let $h : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be measurable such that $\mathbb{E}|h(X, Y)| < \infty$. Then

$$\mathbb{E}[h(X, Y) | Y = y] = \int_{\mathbb{R}} h(x, y) f_{X|Y}(x | y) \, dx, \text{ for } \mathbb{P}^Y\text{-almost all } y \in \mathbb{R}.$$

▷ 21.01.2020

6.4 Introduction to Martingales

Definition 6.5 (Discrete stochastic process)

The family of random variables $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}$ is called a stochastic process.

Definition 6.6 (Discrete filtration)

Let (Ω, \mathcal{A}) be a measurable space. A sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ of sub σ -fields of \mathcal{A} over Ω is called a filtration over (Ω, \mathcal{A}) .

Definition 6.7 (Adapted process)

A stochastic process $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}$ is adapted to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) if for all $n \in \mathbb{N}$, X_n is \mathcal{F}_n - \mathcal{E} -measurable.

Definition 6.8 (Discrete martingale)

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)_{n \in \mathbb{N}}$ a filtration over (Ω, \mathcal{A}) . The random variables $(X_n)_{n \in \mathbb{N}}$ are called a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$: \iff

1. $X_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P})$ and
2. $\mathbb{E}(X_{n+1} | \mathcal{F}_n) = X_n$ \mathbb{P} -almost surely for all $n \in \mathbb{N}$.

Remark 6.8 In the above situation, if instead of (2.) only

$$\mathbb{E}(X_{n+1} | \mathcal{F}_n) \geq X_n \text{ } \mathbb{P}\text{-almost surely for all } n \in \mathbb{N}$$

holds, then we refer to it as a submartingale and if only

$$\mathbb{E}(X_{n+1} | \mathcal{F}_n) \leq X_n \text{ } \mathbb{P}\text{-almost surely for all } n \in \mathbb{N}$$

holds, we call it a supermartingale.

Definition 6.9 (Predictable process)

▷ 23.01.2020

A stochastic process $C_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}_{\geq 2}$ is called predictable with respect to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) if C_{n+1} is \mathcal{F}_n - \mathcal{E} -measurable for all $n \in \mathbb{N}$.

Definition 6.10 (Martingale transform)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})), n \in \mathbb{N}$ be a martingale with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) and let $C_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})), n \in \mathbb{N}_{\geq 2}$ be a predictable process with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$. We define

$$(C \cdot X)_n := \sum_{k=1}^n C_{k+1}(X_{k+1} - X_k)$$

and call $((C \cdot X)_{n \in \mathbb{N}})$ the martingale transform of $(X_n)_{n \in \mathbb{N}}$ by $(C_n)_{n \in \mathbb{N}_{\geq 2}}$.

Lemma 6.3 For martingales $(X_n)_{n \in \mathbb{N}}$ we have $\mathbb{E}X_n = \mathbb{E}X_1$ for all $n \in \mathbb{N}$.

Theorem 6.7 Let $C_{n+1}, X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})), n \in \mathbb{N}$ be stochastic processes, where $(C_n)_{n \in \mathbb{N}_{\geq 2}}$ is predictable and $(X_n)_{n \in \mathbb{N}}$ is a martingale with respect to a filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) . Then $((C \cdot X)_{n \in \mathbb{N}})$ is a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$ with $\mathbb{E}(C \cdot X)_n = 0$ for all $n \in \mathbb{N}$.

6.5 Martingales and Stopping Times

Definition 6.11 (Stopping time)

Let $(\mathcal{F}_n)_{n \in \mathbb{N}}$ be a filtration over a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A random variable $\tau : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{N}_{\leq \infty}, 2^{\mathbb{N}_{\leq \infty}})$ such that

$$\{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}$$

is called a stopping time with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$.

Remark 6.9 In the above setting, τ is a stopping time $\iff \{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$.

Definition 6.12 (Stopped process)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (E, \mathcal{E}), n \in \mathbb{N}$ be a process adapted to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) and let $\tau : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{N}_{\leq \infty}, 2^{\mathbb{N}_{\leq \infty}})$ be a stopping time with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$. We call

$$(X_n^\tau)_{n \in \mathbb{N}}, \text{ where } X_n^\tau(\omega) := X_{\min\{\tau(\omega), n\}}(\omega), \text{ for all } \omega \in \Omega, n \in \mathbb{N}$$

the process $(X_n)_{n \in \mathbb{N}}$ stopped at time τ .

Theorem 6.8 (Elementary stopping theorem)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) and $\tau : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{N}_{\leq \infty}, 2^{\mathbb{N}_{\leq \infty}})$ be a stopping time with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$. Then the stopped process $(X_n^\tau)_{n \in \mathbb{N}}$ is again a martingale with $\mathbb{E}X_n^\tau = \mathbb{E}X_1$ for all $n \in \mathbb{N}$.

Definition 6.13 (σ -algebra of the τ -past)

Let $\tau : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{N}_{\leq \infty}, 2^{\mathbb{N}_{\leq \infty}})$ be a stopping time with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) . We call

$$\mathcal{A}_\tau := \{A \in \mathcal{A} \mid A \cap \{\tau \leq n\} \in \mathcal{F}_n \text{ for all } n \in \mathbb{N}\}$$

the σ -algebra of the τ -past.

Theorem 6.9 Let $\tau : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{N}_{\leq \infty}, 2^{\mathbb{N}_{\leq \infty}})$ be \mathbb{P} -almost surely finite and a stopping time with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) . Let further $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R})), n \in \mathbb{N}$ be a stochastic process adapted to $(\mathcal{F}_n)_{n \in \mathbb{N}}$. Then the random variable

$$X_\tau : \Omega \rightarrow \mathbb{R}, \omega \mapsto X_\tau(\omega) := \begin{cases} X_{\tau(\omega)} & , \text{ if } \tau(\omega) < \infty \\ 0 & , \text{ otherwise} \end{cases}$$

is \mathcal{A}_τ - $\mathcal{B}(\mathbb{R})$ -measurable.

Theorem 6.10 (Doob's stopping theorem)

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) and $\tau : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{N}_{\leq \infty}, 2^{\mathbb{N}_{\leq \infty}})$ be a stopping time with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$. Then X_τ is $\mathbb{P}|_{\mathcal{A}_\tau}$ -integrable and $\mathbb{E}X_\tau = \mathbb{E}X_0$ if either of

1. τ is \mathbb{P} -almost surely bounded,
2. τ is \mathbb{P} -almost surely finite and $(X_n)_{n \in \mathbb{N}}$ is \mathbb{P} -almost surely bounded or
3. $\mathbb{E}\tau < \infty$ and $(|X_{n+1} - X_n|)_{n \in \mathbb{N}}$ is \mathbb{P} -almost surely bounded

is satisfied.

Remark 6.10 If in the above situation $(X_n)_{n \in \mathbb{N}}$ is only a supermartingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}}$ and (1.) - (3.) or

4. τ is \mathbb{P} -almost surely finite and $(X_n)_{n \in \mathbb{N}}$ is \mathbb{P} -almost surely non-negative

then X_τ is $\mathbb{P}|_{\mathcal{A}_\tau}$ -integrable and instead $\mathbb{E}X_\tau \leq \mathbb{E}X_0$.

▷ 28.01.2020

6.6 Martingales and Convergence

Definition 6.14 (L^p -boundedness)

Let $(X_n)_{n \in \mathbb{N}} \in L^p(\Omega, \mathcal{A}, \mathbb{P})$ and let $p \in [1, \infty)$. We say

$$(X_n)_{n \in \mathbb{N}} \text{ is } L^p\text{-bounded} \iff \exists K \in \mathbb{R} : \forall n \in \mathbb{N} : \|X_n\|_p < K.$$

Lemma 6.4 (Upcrossing lemma)

▷ 30.01.2020

Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be a supermartingale with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) and let $a, b \in \mathbb{R}$, $a < b$. Then

$$\forall N \geq 0 : (b - a)\mathbb{E}U_N[a, b] \leq -\mathbb{E} \min\{(X_N - a), 0\},$$

where

$$U_N[a, b] := \max \left\{ k \in \mathbb{N}_0 \mid \begin{array}{l} \exists 0 \leq s_1 < t_1 < s_2 < t_2 < \dots < s_k < t_k \leq N \in \mathbb{N} \\ \text{s.t. } X_{s_i} < a \text{ and } X_{t_i} > b, \forall i \in \{1, \dots, k\} \end{array} \right\}.$$

Lemma 6.5 Let $(X_n)_{n \in \mathbb{N}} \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ be a L^1 -bounded supermartingale with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) . Then the monotone limit

$$U[a, b] := \lim_{N \rightarrow \infty} U_N[a, b]$$

satisfies

$$(b - a)\mathbb{E}U[a, b] \leq |a| + \sup_{n \geq 0} \mathbb{E}|X_n| < \infty.$$

In particular,

$$\mathbb{P}(U[a, b] = \infty) = 0.$$

Theorem 6.11 (Doob's martingale convergence theorem)

Let $(X_n)_{n \in \mathbb{N}} \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ be a L^1 -bounded supermartingale with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) . Then a random variable $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ exists such that $X_n \xrightarrow{\text{a.s.}} X$.

Remark 6.11 Let $(X_n)_{n \in \mathbb{N}} \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ be any supermartingale with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) . If $(X_n)_{n \in \mathbb{N}}$ non-negative \mathbb{P} -almost surely then $(X_n)_{n \in \mathbb{N}}$ is already L^1 -bounded.

Definition 6.15 Random variables $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ are called \mathbb{P} -uniformly integrable if

$$\forall \varepsilon > 0 : \exists K > 0 : \mathbb{E}(\mathbb{1}_{\{|X_n| > K\}} |X_n|) \leq \varepsilon, \forall n \in \mathbb{N}.$$

Remark 6.12 In the above situation if $(X_n)_{n \in \mathbb{N}} \in L^p(\Omega, \mathcal{A}, \mathbb{P})$ for some $p > 1$, they are \mathbb{P} -uniformly integrable as well.

Theorem 6.12 Let $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \rightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $n \in \mathbb{N}$ be a \mathbb{P} -uniformly integrable martingale with respect to filtration $(\mathcal{F}_n)_{n \in \mathbb{N}}$ over (Ω, \mathcal{A}) . Then a random variable $X \in L^1(\Omega, \mathcal{A}, \mathbb{P})$ exists such that

$$X_n \xrightarrow{\text{a.s. \& } L^1} X \quad \text{and} \quad \forall n \in \mathbb{N} : \mathbb{E}(X \mid \mathcal{F}_n) = X_n \text{ } \mathbb{P}\text{-almost surely.}$$